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Non-Thermal Aspects of Black Hole Radiance

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Abstract

The phenomenon of black hole thermodynamics raises several deep issues which any proper theory of quantum gravity must confront: to what extent does the inclusion of the back-reaction alter the thermal character of the radiation, how can the entropy be understood from a microscopic standpoint, what is the ultimate fate of an evaporating black hole, and is the outcome reconcilable with unitary time evolution in quantum mechanics?

In the first part of this thesis, we address the issue of determining what the actual emission spectrum from a black hole is, once the gravitational field of the emitted quanta is included in a quantum mechanical manner. To make the problem tractable, we employ two important approximations: we quantize only the s-wave sector of the full theory, and we consider only single particle emission. By proceeding in the framework of a Hamiltonian path integral description of this system, we are able to integrate out the gravitational field, thereby obtaining an effective action depending only on the matter degrees of freedom. This effective action can then be second quantized in terms of new, corrected, mode solutions thus enabling the calculation of the emission spectrum from modified Bogoliubov coefficients. The results are particularly interesting in the case of emission from Reissner-Nordstrom black holes, since in the extremal limit our results are dramatically different from what a naive, and incorrect, semi-classical calculation would yield.

The other major topic which we discuss is the dynamics of quantum fields on background geometries which undergo quantum tunneling. An example of such a system which has important implications for both cosmology and quantum gravity in general, is the tunneling of a false vacuum bubble leading to the creation of a new universe. To determine what the state of a scalar field would be as a result of such a process, we make a WKB approximation of the Wheeler-DeWitt equation to obtain an imaginary time evolution equation for the scalar field. The state after tunneling is then found by solving this equation on a portion of the Euclidean Kruskal manifold, the properties of which serve to ensure that at late times thermal radiation emerges.

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Contents

Chapter 1

Introduction

Perhaps the most conspicuous deficiency in a physicist's current understanding of nature is the absence of a usable quantum mechanical theory of gravity. Standing in the way has been an utter lack of experimental data to serve as a guide toward such a theory. On one hand, the Standard Model of particle physics, while not without unresolved issues of its own, provides a coherent description of all phenomena which have been observed at existing accelerators; on the other hand, string theory, while remaining a viable candidate for a theory of everything, is presently incapable of making further predictions which can be tested by these same accelerators. In the face of such a situation there is still reason to believe that progress can be made by extracting clues from the structure of the theories that we know today. The remarkable quantum mechanical properties of black holes are an important example – the phenomenon of black hole thermodynamics is tantalizingly suggestive of a more comprehensive theory.

Before discussing the relevance of black holes to quantum gravity, which is the focus of the present work, it will be useful to recall some of the development of the classical theory of black holes. Black holes are hard to see. For the experimentalist, this is true for obvious reasons: the defining property of black holes as regions of spacetime from which nothing can emerge means that their existence can only be inferred, either through their gravitational effects on neighboring bodies, or by viewing the characteristic glow of matter as it is sucked into a hole. Nevertheless, recent observations have essentially removed any doubt as to the existence of black holes in our universe. For many years, theorists were similarly unable to see black holes in the equations of general relativity, although in this case the difficulty was due to faulty vision rather than a lack of evidence. Although Schwarzschild wrote down the metric describing the geometry outside a spherically symmetric body,

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

immediately after the discovery of the field equations, an incomplete understanding of the geometry's global structure led to the erroneous conclusion that a singularity occurs at the horizon, $r = 2M$, rendering the solution for $r \leq 2M$ unphysical. Only much later was it realized that the singularity was merely a coordinate artifact, and that physical quantities are entirely well behaved at the horizon. With this came the understanding that there exists a *true* singularity at $r = 0$, but it was suspected that deviations from spherical symmetry would cause the singularity to be smoothed out in a realistic collapse process. However, once Penrose [1] proved that singularities *do* occur in classical relativity under very generic conditions, the modern age of black hole research was begun.

Two theorems in classical relativity served to presage the subsequent development of black hole thermodynamics. First, it was found that the mass and angular momentum of black holes obey the so-called "first law of black hole thermodynamics":

$$dM = \frac{1}{8\pi}\kappa dA + \Omega dJ,$$

J being the hole's angular momentum, A and Ω the horizon's area and angular velocity. κ is the surface gravity, defined as the force which a person at infinity would need to exert in order to keep an object suspended at the horizon. For the Schwarzschild hole, $\kappa = 1/4M$. Second, it was found that in any process, the total area of black hole horizons necessarily increases — the so-called "second law of black hole thermodynamics". These results led Bekenstein [2] to suggest that a black hole possessed an entropy proportional to A, and that the conventional second law of thermodynamics would be upheld when this entropy was taken into account. While the formulae suggest that something proportional to κ plays the role of temperature, this point was entirely obscure from the standpoint of classical relativity, since in this context a black hole can't emit anything, much less thermal radiation.

At this point, quantum mechanics enters the game: Hawking [3] showed that quantum effects cause a black hole to emit exactly thermal radiation with a temperature $T = \kappa/2\pi$. This result immediately leads to the identification $S = A/4$. As we shall consider the derivation of this effect in detail later, let us simply remark here that it is intimately connected to the existence of the horizon as a surface of infinite redshift. This allows particles of negative energy to exist inside the horizon, leading to the possibility of particle-antiparticle creation, one particle flowing into the hole, one flowing out, without violating conservation of energy. Perhaps the most startling implication of this result is that, although it only relies on fairly conservative assumptions about the interaction of quantum fields with gravity, it leads to a fundamental conflict with the usual formulation of quantum mechanics. For if the black hole continually emits radiation it will eventually disappear altogether, and since thermal radiation is uncorrelated, there will not be any trace left of the matter which originally formed the hole. More concisely, the process is not described by by unitary evolution since any initial state forming the black hole leads to the same thermal density matrix after the hole has evaporated. If this is true, then quantum mechanics will have to be revised in order to be compatible with this phenomenon of information loss. On the other hand, this conclusion may be premature, as the approximation employed in computing the radiance breaks down once the hole becomes sufficiently small, so that there is in fact no compelling reason why the black hole has to disappear. Various alternatives to the information loss scenario have been the subject of much discussion recently (two reviews, written from quite different viewpoints, are [4, 5]) but no consensus has been established. Much of the recent work has been done in the context of two dimensional models [6] which, owing to conformal invariance, allow more analytical progress than was previously possible. Unfortunately, the key questions remain unanswered.

The other closely related problem is to obtain a proper understanding of the formula $S = A/4$. It seems quite likely that a resolution of the information loss puzzle is contingent upon understanding in what sense, if any, a black hole has the number of states $e^{A/4}$. One possibility is that a black hole has this number of weakly coupled states localized near the horizon, proper accounting of which will lead to correlations in the outgoing radiation. Although the radiation may look approximately thermal, the correlations could be sufficient to encode all details of the initial state. Any computation of such correlations must go beyond the free field approximation originally employed by Hawking; with this goal in mind, a large part of the present work is devoted to obtaining the leading corrections to the free field results.

In the course of this work we shall be discussing these problems mainly in the context of the semiclassical approximation. However, we should point out that the validity of this approximation is by no means assured, due to the peculiar properties of the horizon. Indeed, it has been argued that strong coupling effects [7] and quantum fluctuations [8] make this description unreliable.

The remainder of this thesis is organized as follows. In Chapter 2 we develop the Hamiltonian formulation of gravity, which will play a key role in the material that follows. We consider both classical and quantum aspects, paying special attention to the spherically symmetric case. Chapter 3 provides an overview of quantum field theory in curved space, and a detailed discussion of black hole radiance. The discussion is phrased in terms of a nonstandard coordinate system for the black hole geometry [9], which is particularly convenient for doing computations near the horizon. This coordinate system also provides insights into the global structure of the geometry, as we discuss. Finally, the role of fluctuations in the stress-energy is considered, with emphasis on the impact these effects have on the moving mirror model. In Chapter 2 we turn to the main focus of this thesis: the calculation of self-interaction corrections to the black hole emission spectrum [10, 11]. Drawing upon material developed in the previous chapters, we show how these corrections can be deduced by calculating the effective action for a gravitating thin shell. We then discuss the relation between the first and second quantized approaches to this problem, and finally, the difficulty in obtaining multi-particle correlations by a straightforward extension of the single particle calculation. In Chapter 5 we describe some attempts at gaining a microscopic understanding of the black hole entropy by counting fluctuations of the geometry. As we will see, the problem is not how to obtain a sufficiently large number of states, but rather how to control the divergences which inevitably occur. The topic of Chapter 6 is the effect of quantum tunneling upon radiance phenomena [12]. The main example concerns black hole formation via the tunneling of a false vacuum bubble, a process which nicely illustrates the interplay between the Hamiltonian and Lagrangian approaches to quantum gravity. We will show how thermal radiation emerges from the black hole at late times, even though the standard calculation of the radiance is inapplicable.

Chapter 2

Hamiltonian Formulation of Gravity

2.1 Classical Theory

To proceed with the Hamiltonian approach to gravity we perform a $3+1$ decomposition of the spacetime manifold, singling out a time coordinate. A Hamiltonian is then identified that propagates the three geometry along the time direction. The subtlety involved in this procedure is caused by the reparameterization invariance of the Einstein-Hilbert action: due to the existence of redundant variables, time evolution is not uniquely defined unless a gauge is specified. However, there is a well developed formalism for dealing with such systems [13], as will now be discussed.

The key to separating out the physical and redundant degrees of freedom is to write the metric in $3+1$ form as [14]

$$ds^2 = -(N^t dt)^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.1}$$

With this labelling, the Einstein-Hilbert action becomes:

$$\mathcal{L}_G = \frac{1}{16\pi}\sqrt{-g}\,\mathcal{R} = \frac{1}{16\pi}\sqrt{h}\,N^t\,[{}^3\mathcal{R} + K_{ij}K^{ij} - K^2], \tag{2.2}$$

where ${}^3\mathcal{R}$ is the Ricci scalar associated with h_{ij}, K_{ab} is the extrinsic curvature of a constant t hypersurface:

$$K_{ab} = \frac{1}{2N^t}[\dot{h}_{ij} - N_{i|j} - N_{j|i}], \tag{2.3}$$

and $K \equiv K^a_a$. Throughout, Latin indices are raised and lowered by h_{ij}, and $|$ denotes covariant differentiation with respect to h_{ij}.

Now, it is seen that no time derivatives of N^t or N^i appear in the gravitational action, nor will they when matter is included, so that the canonical momenta conjugate to these variables vanish identically:

$$\pi_{N^t} \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^t} = 0 \quad ; \quad \pi_{N^i} \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0. \tag{2.4}$$

These are constraints on the phase space. h_{ij}, on the other hand, have nonvanishing canonical momenta π_{ij}. These can be computed and used to put $\mathcal{L}_G$ in canonical form:

$$\mathcal{L}_G = \pi_{ij}\dot{h}^{ij} - N^t\mathcal{H}_t - N_i\mathcal{H}^i \tag{2.5}$$

where

$$\mathcal{H}_t = 8\pi h^{-\frac{1}{2}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})\pi^{ij}\pi^{kl} - \frac{1}{16\pi}h^{\frac{1}{2}}\,{}^3\mathcal{R}, \tag{2.6}$$

$$\mathcal{H}^i = -2\pi^{ij}_{|j}. \tag{2.7}$$

The constraints $\pi_{N^t} = \pi_{N^i} = 0$ must hold at all times, which means that their Poisson brackets with the Hamiltonian must vanish. This condition yields the secondary constraints

$$\mathcal{H}_t = \mathcal{H}^i = 0. \tag{2.8}$$

All the dynamics of gravity is contained in these constraints. It should be stressed that these constraints hold "weakly" — they are to be imposed only after all Poisson brackets have been computed.

2.1.1 Surface Terms

In deriving the canonical form of the action we ignored all surface terms which arose. However, for gravity in asymptotically flat space surface terms play an important role and cannot be neglected, so we write

$$H_G = \int d^3x[N^t\mathcal{H}_t + N_i\mathcal{H}^i] + \text{surface terms}. \tag{2.9}$$

Regge and Teitelboim [15] showed that Einstein's equations are equivalent to Hamilton's equations applied to H_G only for a particular choice of surface terms. The point is that if Einstein's equations are written as

$$\dot{h}_{ij} = A_{ij}(h,\pi) \quad ; \quad \dot{\pi}_{ij} = -B_{ij}(h,\pi) \tag{2.10}$$

then we need the variation of H_G to be

$$\delta H_G = A_{ij}\delta\pi^{ij} + B_{ij}\delta h^{ij} \tag{2.11}$$

to ensure consistency with Einstein's equations. To put δH_G in this form we must integrate by parts, because space derivatives of h_{ij} and π_{ij} appear in H_G, and then demand that the resulting surface term cancels with the one we have included in (2.9). In [15] the surface terms were worked out assuming a particular rate of fall-off of h_{ij}, π_{ij}, N^t, and N^i at infinity. This condition can be relaxed [16], and in general the full Hamiltonian is:

$$H_G = \int_\Sigma d^3x[N^t\mathcal{H}_t + N_i\mathcal{H}^i] + \int_{\partial\Sigma} ds_l[\mathcal{H}^l_{1s} + \mathcal{H}^l_{2s}] \tag{2.12}$$

with

$$\begin{aligned} H^l_{1s} &= \frac{1}{16\pi}(N^t\sqrt{h}h^{ij}U^l_{ij} + 2\pi^{li}N_i) \\ H^l_{2s} &= \frac{1}{16\pi}\left[\frac{1}{N^t}\sqrt{h}(N^i\partial_i N^l - N^l\partial_i N^i)\right] \end{aligned} \tag{2.13}$$

where

$$U^i_{jk} = \Gamma^i_{jk} - \delta^i_{(j}\Gamma^l_{\ k)}. \tag{2.14}$$

The result of Regge and Teitelboim corresponds to choosing fall-off conditions such that H^l_{2s} vanishes at infinity.

Since $\mathcal{H}_t$ and $\mathcal{H}^i$ vanish when the constraints are satisfied, the numerical value of the Hamiltonian is given solely by the surface terms. This numerical value is the ADM mass of the system.

2.1.2 Spherical Symmetry

We now specialize to the case of spherically symmetric geometries [17, 18, 19]. It is possible to proceed substantially further in this case because there are no propagating degrees of freedom – after solving the constraints there will only be one free parameter remaining: the ADM mass. The metric is written

$$ds^2 = -(N^t dt)^2 + L^2(dr + N^r dt)^2 + R^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{2.15}$$

and the action is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,\mathcal{R} = \int dt\,dr\,[\pi_R\dot{R} + \pi_L\dot{L} - N^t\mathcal{H}_t - N^r\mathcal{H}_r] \tag{2.16}$$

with

$$\mathcal{H}_t = \frac{L\pi_L^2}{2R^2} - \frac{\pi_L\pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} \quad ; \quad \mathcal{H}_r = R'\pi_R - L\pi_L' \tag{2.17}$$

where $'$ represents d/dr.

As before, the Hamiltonian is

$$H = \int dr[N^t\mathcal{H}_t + N^r\mathcal{H}_r] + M \tag{2.18}$$

and the constraints are

$$\mathcal{H}_t = \mathcal{H}_r = 0. \tag{2.19}$$

The constraints can be solved as follows. π_R is eliminated by forming the linear combination of constraints

$$\frac{R'}{L}\mathcal{H}_t + \frac{\pi_L}{RL}\mathcal{H}_r = 0.$$

Defining

$$\mathcal{M}(r) \equiv \frac{\pi_L^2}{2R} + \frac{R}{2} - \frac{RR'^2}{2L^2}, \tag{2.20}$$

this constraint is equivalent to $\mathcal{M}' = 0$. By comparing with (2.13) it can be shown that $\mathcal{M}(\infty)$ is the ADM mass. The constraints can now be solved for the momenta:

$$\pi_L = \eta R\sqrt{(R'/L)^2 - 1 + 2M/R} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L' \tag{2.21}$$

where $\eta = \pm 1$.

Since we have defined the new variable, $\mathcal{M}(r)$, it is natural to ask what the momentum conjugate to $\mathcal{M}(r)$ is. From the fundamental Poisson brackets,

$$\{L(r), \pi_L(r')\} = \{R(r), \pi_R(r')\} = \delta(r - r') \tag{2.22}$$

it is straightforward to check that

$$\pi_{\mathcal{M}} \equiv \frac{L\sqrt{(R'/L)^2 - 1 + 2\mathcal{M}/R}}{1 - 2\mathcal{M}/R} \tag{2.23}$$

satisfies

$$\{\mathcal{M}(r), \pi_{\mathcal{M}}(r')\} = \delta(r - r'). \tag{2.24}$$

$\pi_{\mathcal{M}}$ has a simple physical interpretation. Consider the geometry in Schwarzschild coordinates:

$$ds^2 = -(1 - 2M/r_s)dt_s^2 + \frac{dr_s^2}{1 - 2M/r_s} + r_s^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.25}$$

and then transform to new coordinates $t(t_s, r_s)$, $r(t_s, r_s)$. One finds that in the new coordinates,

$$R = r_s(t, r) \qquad ; \qquad L^2 = \frac{1}{1 - 2M/r_s}\left(\frac{\partial r_s}{\partial r}\right)^2 - (1 - 2M/r_s)\left(\frac{\partial t_s}{\partial r}\right)^2 \tag{2.26}$$

which then yields

$$\pi_{\mathcal{M}}(r) = \frac{\partial t_s}{\partial r}. \tag{2.27}$$

In other words, $\pi_{\mathcal{M}}$ measures the rate at which Schwarzschild time changes along a hypersurface. $P_M \equiv \int \pi_{\mathcal{M}}(r)dr$ is conjugate to the ADM mass: $\{M, P_M\} = 1$. This gives clear expression to the often heard statement that the energy generates time translations at infinity.

Since P_M is related to the behavior at infinity, it is not surprising that it is invariant only under small diffeomorphisms:

$$\delta P_M = \left\{\int dr[f(r)\mathcal{H}_t + g(r)\mathcal{H}_r]\, , P_M\right\}$$

vanishes provided

$$f(\infty), g(\infty) \leq \left.\frac{\pi_L}{RL}\right|_{r=\infty}$$

as can be checked explicitly. For most coordinate choices, the inequality reduces to $f(\infty) = g(\infty) = 0$.

2.2 Quantization

Once the Hamiltonian formulation of gravity has been developed, the transition to the quantum theory is quite straightforward. This is true only in a formal sense, though, since operator ordering ambiguities will be left unresolved, as will the problem of ultraviolet divergences. Furthermore, the interpretation of the resulting theory is not at all clear, as we will briefly discuss. More detailed discussion of interpretational issues can be found in [20].

To quantize, we realize the phase space variables h_{ij}, π_{ij} as operators satisfying the commutation relations

$$[h_{ij}(x^i), \pi^{mn}(x'^i)] = i\delta_i^m \delta_j^n \delta^3(x^i - x'^i) \tag{2.28}$$

We therefore make the replacements,

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}}.$$

The states of the system are then described by functions of the three-geometry, *i.e.* by the wave functionals $\Psi[h_{ij}(x^i)]$. Physical states are required to be annihilated by the constraints:

$$\mathcal{H}^i \Psi[h_{ij}] = 2i \left(\frac{\delta}{\delta h_{ij}}\right)_{|j} \Psi[h_{ij}] = 0 \tag{2.29}$$

$$\mathcal{H}_t \Psi[h_{ij}] = -\left[\frac{8\pi}{m_p^2} h^{\frac{1}{2}} G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \frac{m_p^2}{16\pi} h^{\frac{1}{2}}\, {}^3\mathcal{R}\right] \Psi[h_{ij}] = 0 \tag{2.30}$$

where $G_{ijkl} = h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}$, and we have restored the Planck mass. The first set of constraints simply says that physical states should be invariant under reparameterizing the spacelike hypersurfaces; they are analogous to Gauss' law as they are linear in derivatives. The final constraint is quadratic in derivatives and is known as the Wheeler-Dewitt equation. In asymptotically flat space, the wavefunction also satisfies a non-trivial Schrodinger equation:

$$H\Psi[h_{ij}, t] = -i\frac{\partial \Psi[h_{ij}, t]}{\partial t}. \tag{2.31}$$

It is then assumed that $\Psi^*[h_{ij}]\Psi[h_{ij}]$ can somehow be interpreted as a probability density over three-geometries.

Even if we knew the correct operator ordering prescription, the full Wheeler-Dewitt equation would still be much too difficult to solve. However, it can be simplified greatly by employing the WKB approximation, which amounts to taking $m_p \to \infty$. In the WKB approximation the wavefunction is written in the form

$$\Psi[h_{ij}] = e^{im_p^2 S[h_{ij}]}. \tag{2.32}$$

Inserting this expression into (2.30) and keeping only terms of order m_p^2, we find that S satisfies:

$$16\pi G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} - \frac{1}{16\pi}h^{\frac{1}{2}}\,{}^3\mathcal{R} = 0. \tag{2.33}$$

This is the Einstein-Hamilton-Jacobi equation, which means that $S[h_{ij}]$ is the classical action associated with a solution to Einsteins's equations. Specifically, given some solution $g_{\mu\nu}(x^\mu)$, $S[h_{ij}]$ is found by integrating $S_G = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,\mathcal{R}$ in the region between some reference hypersurface and the hypersurface $h_{ij}(x^i)$. Thus the WKB approximation reduces the task of solving a functional differential equation to solving a set of partial differential equations.

As we remarked earlier, it is difficult to interpret the wavefunction $\Psi[h_{ij}]$. In particular, what does it mean to say that $|\Psi|^2$ is the probability to find some three-geometry? It may well be that the wavefunction only has meaning once matter is included. Then, writing $\Psi[h_{ij},\phi]$, we would interpret

$$\frac{|\Psi[h_{ij},\phi]|^2}{\int D\phi|\Psi[h_{ij},\phi]|^2}$$

as the relative probability of finding various matter configurations on the hypersurface h_{ij}. We will return to this intepretation in later sections when we couple a scalar field to gravity.

2.2.1 Spherical Symmetry

In the spherically symmetric case we have the operators L, R and $\pi_L = -i\frac{\delta}{\delta L}$, $\pi_R = -i\frac{\delta}{\delta R}$, and the wavefunction, $\Psi[L,R]$, which satisfies the constraints

$$\mathcal{H}_t\Psi[L,R] = \mathcal{H}_r\Psi[L,R] = 0. \tag{2.34}$$

In this simplified theory the quadratic constraint is still too difficult to solve so we again employ the WKB approximation and write

$$\Psi[L,R] = e^{iS[L,R]}. \tag{2.35}$$

Since the WKB approximation amounts to the replacements $\pi_L \to \frac{\delta S}{\delta L}$, $\pi_R \to \frac{\delta S}{\delta R}$, and using (2.21), S is found to satisfy

$$\frac{\delta S}{\delta L} = \eta R\sqrt{(R'/L)^2 - 1 + 2M/R}$$
$$\frac{\delta S}{\delta R} = \frac{L}{R'}\frac{d}{dr}\left(\eta R\sqrt{(R'/L)^2 - 1 + 2M/R}\right). \tag{2.36}$$

These expressions are most easily integrated as follows. Start from some arbitrary geometry L, R, and vary L while holding R fixed until $(R'/L)^2 - 1 + 2M/R = 0$. The contribution to S from this variation is

$$\begin{aligned} S[L,R] &= \eta R\int dr\, dL\sqrt{(R'/L)^2 - 1 + 2M/R} \\ &= \eta\int dr\left[RL\sqrt{(R'/L)^2 - 1 + 2M/R}\right. \\ &\left. + RR'\log\left|\frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M/R}}{\sqrt{|1-2M/R|}}\right|\right]. \end{aligned} \tag{2.37}$$

Now vary L, R, while keeping $(R'/L)^2 - 1 + 2M/R = 0$, to some set geometry. There is no contribution to S from this variation, so (2.37) is the complete solution. The wavefunction, Ψ, either oscillates or decays exponentially depending on the sign of $(R'/L)^2 - 1 + 2M/R$. This is also the condition which determines whether the hypersurface L, R, can be embedded in the classical Schwarzschild geometry.

The description of the state in terms of the wavefunction $\Psi[L,R]$ is, of course, highly redundant since, according to the constraints, the wavefunction must take the same value for many different L's and R's. If we wish to eliminate this redundancy we should describe the state in terms of physical observables, *i.e.* those which commute with the constraints. In the present case there are only two such observables [19, 21] : the ADM mass, M, and its canonical momentum P_M. To satisfy the commutation relation $[M, P_M] = i$

we make the replacement $M = i\partial/\partial P_M$, so that a state of definite mass is written

$$\Psi_M(P_M) = e^{-iMP_M}. \tag{2.38}$$

If the time dependence is also included we have

$$\Psi_M(P_M, t) = e^{-iM(P_M+t)}. \tag{2.39}$$

It is seen that P_M functions as an intrinsic time variable.

The emergence of an intrinsic time in this system is quite interesting in light of the so-called "problem of time" in quantum gravity. In the asymptotically flat case this problem is relatively benign, since the wavefunction depends on the time t, which can be measured by clocks at infinity. However, for a closed universe there is no such asymptotic region and no obvious variable to play the role of time. This makes the wavefunction of a closed universe especially difficult to interpret. In quantum mechanics, we interpret the wavefunction $\psi(x,t)$ by saying that at fixed time t the probability to find x is $|\psi(x,t)|^2$. To carry this procedure over to the quantum theory of a closed universe we need to specify a variable which can be fixed in order to determine probabilities. In the spherically symmetric, asymptotically flat case, P_M, like t, can play this role, but no such variable is known in general.

This completes our discussion of the quantization of the pure gravity theory. In what follows we turn to the effects of incorporating matter into the system.

Chapter 3

Quantum Field Theory Near Black Holes

3.1 A Useful Coordinate System

Calculations of physical effects in the presence of black holes are greatly simplified by employing appropriate coordinates. In this section, we will describe the properties of a little known set of coordinates [9] which will be used repeatedly in the material that follows.

Schwarzschild found his remarkable exact solution for the geometry outside a star in general relativity quite soon after Einstein derived the field equations. Further study of this geometry over the course of several decades revealed a series of surprises: the existence and physical relevance of pure vacuum "black hole" solutions; the incompleteness of the space-time covered by the original Schwarzschild coordinates, and the highly non-trivial global structure of its completion; and the dynamic nature of the physics in this geometry despite its static mathematical form, revealed perhaps most dramatically by the Hawking radiance [3]. Discussions of this material can now be found in advanced textbooks [24], but they are hardly limpid.

In the course of investigating an improvement to the standard calculation of this radiance to take into account its self-gravity, as we detail in later sections, we came upon a remarkably simple form for the line element of Schwarzschild (and Reissner-Nordstrom) geometry. This line element has an interesting history [22], but as far as we know it has never been discussed from a modern point of view. We have found that several of the more subtle features of the geometry become especially easy to see when this line element is used.

To motivate the form of the coordinates we reconsider the constraint equations of spherically symmetric gravity (2.17):

$$\mathcal{H}_t = \frac{\pi_L^2}{2R^2} - \frac{\pi_L \pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} = 0,$$
$$\mathcal{H}_r = R'\pi_R - L\pi' = 0. \tag{3.1}$$

The canonical momenta are given by

$$\pi_L = \frac{N^r R'}{N^t} - \frac{R\dot{R}}{N^t} \quad ; \quad \pi_R = \frac{(N^r LR)'}{N^t} - \frac{(\dot{LR})}{N^t}. \tag{3.2}$$

We can arrive at a particular set of coordinates by choosing a gauge and solving the constraints. Our choice is simply $L = 1$, $R = r$. With this choice the equations simplify drastically, and one easily solves to find $\pi_L = \sqrt{2Mr}$, $\pi_R = \sqrt{\frac{M}{2r}}$ and then $N^t = \pm 1$, $N^r = \pm\sqrt{\frac{2M}{r}}$. Thus for the line element we have

$$ds^2 = -dt^2 + (dr \pm \sqrt{\frac{2M}{r}} dt)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{3.3}$$

M, which appears as an integration constant, of course is to be interpreted as the mass of the black hole described by this line element.

For the Reissner-Nordstrom geometry, the same gauge choice leads to a metric of the same form, with the only change that $2M \to 2M - Q^2/r$.

These line elements are stationary – that is, invariant under translation of t, but not static – that is, invariant under reversal of the sign of t. Indeed reversal of this sign interchanges the $\pm$ in (3.3), a feature we will interpret further below. Another peculiar feature is that each constant time slice $dt = 0$ is simply flat Euclidean space!

We can obtain a physical interpretation of these coordinates by comparing them to those of Lemaitre [23], in terms of which the line element reads

$$ds^2 = -d\tau^2 + \frac{(2M)^{2/3}}{[\frac{3}{2}(r_L + \tau)]^{2/3}} dr_L^2 - (2M)^{2/3}[\frac{3}{2}(r_L - \tau)]^{4/3}(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4}$$

As the Lemaitre coordinates are synchronous ($g_{\tau\tau} = -1$, $g_{\tau\mathrm{i}} = 0$), a class of timelike geodesics is given by motion along the time lines ($r_L, \theta, \phi =$constant), and the proper time along the geodesics is given by the coordinate τ. To arrive at (3.4) we retain the Lemaitre time coordinate, but now demand that the radial coordinate squared be the coefficient multiplying $d\theta^2 + \sin^2\theta d\phi^2$. In other words, we write

$$t = \tau \qquad ; \qquad r = (2M)^{1/3}[\frac{3}{2}(r_L - \tau)]^{2/3}. \tag{3.5}$$

A simple calculation then leads from (3.4) to (3.3) with the upper choice of sign; the lower choice is obtained by repeating the same steps starting from Lemaitre coordinates with the sign of τ reversed.

Finally, let us note that these coordinates are related to Schwarzschild coordinates,

$$ds^2 = -(1 - \frac{2M}{r})dt_s^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{3.6}$$

by a change of time slicing,

$$t_s = t - 2\sqrt{2Mr} - 2M\log\left[\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right]. \tag{3.7}$$

In contrast to the surfaces of constant t_s, the constant t surfaces pass smoothly through the horizon and extend to the future singularity free of coordinate singularities.

In terms of r and t, the radially ingoing and outgoing null geodesics are given by

$$\begin{aligned} \text{ingoing:} \quad & t + r - 2\sqrt{2Mr} + 4M\log[\sqrt{r} + \sqrt{2M}] = v = \text{ constant} \\ \text{outgoing:} \quad & t - r - 2\sqrt{2Mr} - 4M\log[\sqrt{r} - \sqrt{2M}] = u = \text{ constant} . \end{aligned} \tag{3.8}$$

3.1.1 Global Structure

Now let us discuss the global properties of our coordinate system. Perhaps the clearest approach to such questions is via consideration of the properties of light rays. Taking for definiteness the upper sign in (3.3), and without any essential loss of generality restricting to the case $d\theta = d\phi = 0$ appropriate to the $\theta = \pi/2$ sections, we find that $ds^2 = 0$ when

$$\frac{dr}{dt} = -\sqrt{\frac{2M}{r}} \mp 1 \ . \tag{3.9}$$

For the class of light rays governed by the upper sign, we can cover the entire range $0 < r < \infty$ as t varies. In particular one meets no obstruction, nor any special structure, at the horizon $r = 2M$. For the class of light rays governed by the lower sign there is structure at $r = 2M$. When $r > 2M$ one has a positive slope for $\frac{dr}{dt}$, and r ranges over $2M < r < \infty$. When $r < 2M$ one has a negative slope for $\frac{dr}{dt}$, and r ranges over $0 < r < 2M$. When $r = 2M$ it does not vary with t. From these properties, one infers that our light rays cover regions I and II in the Penrose diagram, as displayed in Fig. 3.1. Let us emphasize that the properties of the Penrose diagram can be *inferred* from the properties of the light rays, although we will not belabor that point here.

If one chooses instead the lower sign in (3.3) , and performs a similar analysis, one finds that regions I and II′ are covered. Patching these together with the sectors found previously, one still does not have a complete space-time. However our line element is not yet exhausted. For in drawing Figure 1 we have implicitly assumed that t increases along light rays which point up ("towards the future"). Logically, and to maintain symmetry, one should consider also the opposite case, that the coordinate t increases towards the past. By doing this, one generates coordinate systems covering regions I′ and II′ respectively I′ and II′, for the upper and lower signs in (3.3) . Thus the complete Penrose diagram is covered with patches each governed by a stationary – but not static – metric, and with non-trivial regions of overlap.

In the Reissner-Nordstrom case the generalization of (3.3) has a coordinate singularity at $r = Q^2/2M$. However this singularity is inside both horizons, and does not pose a serious obstruction to a global analysis. One obtains the complete Penrose diagram also in this case by iterating constructions similar to those just sketched.

The usual Schwarzschild line element appears to be time reversal symmetric, but when the global structure of the space-time it defines is taken into account one sees that this appearance is misleading. The fully extended light-rays in Figure 1 go from empty space to a singularity as t advances (they pass from region I into region II), which is definitely distinguishable

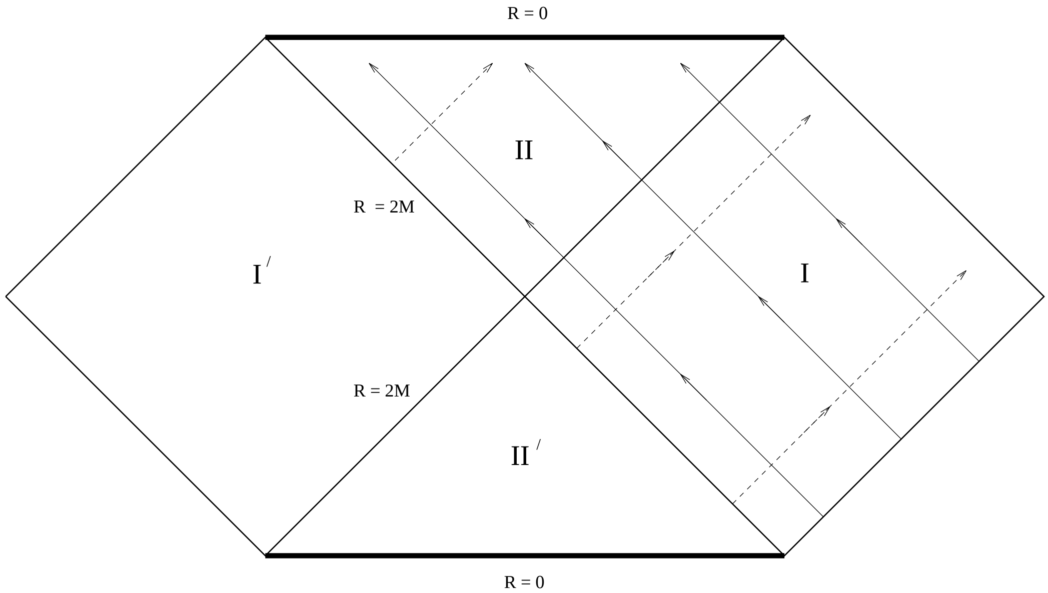

Figure 3.1: Penrose diagram for the Schwarzschild geometry. As described in the text, r and t in one coordinate patch, for the upper sign of the line element, cover regions I and II. As is clear from the diagram, the ingoing light rays are captured in their entirety (into the singularity), whereas the outgoing light rays cannot be traced back past the horizon.

from the reverse process. There is a symmetry which relates these to the corresponding rays going from region II′ to region I′, however it involves not merely changing the sign of t in the Schwarzschild metric, but rather going to a completely disjoint region of the space-time. This actual symmetry of the space-time is if anything more obvious in our construction than in the standard one. Thus by taking the line-element in region I stationary rather than static we have lost some false symmetry while making the true symmetry – and its necessary connection with the existence of region I′ (constructed, as we have seen, by simultaneously reversing the sign of t *and* interchanging the future with the past) – more obvious.

3.2 Quantum Field Theory in Curved Space

In this section we briefly present some of the important results in the theory of quantum fields in curved space. Our goal is primarily to establish notation and to write down some formulas which will be referred to in later sections. Comprehensive reviews of the subject can be found in [24, 25].

Quantum field theory in curved space is a hybrid of quantum and classical field theory, which one hopes reliably describes the behavior of matter in regions of relatively low space-time curvature. The approach is to couple a quantum field to a metric tensor, which is treated as a classical variable. For simplicity, we will consider a massless scalar field with action

$$S = -\frac{1}{2}\int d^4x \sqrt{-g}\, g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi. \tag{3.10}$$

The field satisfies the wave equation,

$$(\sqrt{-g}\, g^{\mu\nu}\phi_{,\mu})_{,\nu} = 0 \tag{3.11}$$

Associated with this wave equation is the conserved inner product

$$(\phi_1, \phi_2) \equiv -i\int_\Sigma d^3x \sqrt{h}\, n^\mu \phi_1(x^i, t)\, \overleftrightarrow{\partial_\mu}\, \phi_2^*(x^i, t) \tag{3.12}$$

where ϕ_1 and ϕ_2 are two solutions of the wave equation. Here Σ is a Cauchy surface, and n^μ is a future pointing unit vector normal to Σ. In order to second quantize the field we must write down a complete set of "positive frequency" solutions, $u_i(x^\mu)$, which satisfy $(u_i, u_j) = \delta_{ij}$. This is where the ambiguity in the quantization process occurs, since there are many different sets of u_i's, which will be shown to lead to inequivalent quantizations. Once we have chosen a set, the field operator is written in terms of creation and annihilation operators:

$$\phi = \sum_i [a_i u_i + a_i^\dagger u_i^*] \tag{3.13}$$

where a, $a^\dagger$ obey

$$[a_i, a_j^\dagger] = \delta_{ij}. \tag{3.14}$$

The vacuum state is defined by $a_i|0_u\rangle = 0$, and particle states are created by applying $a^\dagger$'s to $|0_u\rangle$.

On the other hand, suppose that instead of u_i we had chosen the modes v_i in which to expand the field:

$$\phi = \sum_i [b_i v_i + b_i^\dagger v_i^*]. \tag{3.15}$$

Then the vacuum would be defined by $b_i|0_v\rangle = 0$, and particle states would be created by applying $b^\dagger$'s to $|0_v\rangle$. The obvious question which arises is: what is the relation between the states defined by u_i and those defined by v_i. To answer this, we note that by equating (3.13) and (3.15), and taking inner products, we obtain the "Bogoliubov transformation":

$$b_j = \sum_j (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger) \tag{3.16}$$

where

$$\alpha_{ij} = (v_i, u_j) \quad ; \quad \beta_{ij} = -(v_i, u_j^*) \tag{3.17}$$

The Bogoliubov coefficients obey the completeness relations,

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij}$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0. \tag{3.18}$$

It can then be shown that the states are related by

$$|\psi_u\rangle = C : \exp[\frac{1}{2}a(\alpha^{-1}\beta)a + a(\alpha^{-1} - 1)a^\dagger + \frac{1}{2}a^\dagger(-\beta^*\alpha^{-1})a^\dagger] : |\psi_v\rangle \tag{3.19}$$

where C is a constant. The average number of v particles in the u vacuum is

$$\langle 0_u|b_i^\dagger b_i|0_u\rangle = \sum_j |\beta_{ji}|^2. \tag{3.20}$$

3.3 Black Hole Radiance

Now we turn to the quantization of a massless scalar field in the presence of collapsing matter. The goal is to compute the flux of particles on $\mathcal{J}^+$ given some initial state defined on $\mathcal{J}^-$. To completely determine this flux on all of $\mathcal{J}^+$, one would have to solve the wave equation mode by mode in the region bounded by $\mathcal{J}^-$ and Σ, where Σ is a constant t hypersurface which crosses the horizon to the future of the collapsing matter. This is clearly intractable, since the geometry inside the collapsing matter may be very complicated, and the scalar field might interact with the matter in an arbitrarily complex fashion. Fortunately, all of the dependence of the particle flux on $\mathcal{J}^+$ on these factors dies out at sufficiently late times, and the radiation becomes, at least to leading order, completely independent of the details of the collapse process. Thus we shift our goal to calculating this late-time radiation and to showing that it is indeed universal.

The first important observation is that the geometry is entirely smooth as long as one stays sufficiently far from the singularity at $r = 0$. In particular, for a black hole much larger than m_p, the geometry is smooth near the horizon — in fact, the curvature can be made arbitrarily small by making the black hole arbitrarily massive. This strongly suggests the conlusion that however complicated the state of the field is after propagating through the matter, it should certainly appear nonsingular to inertial observers near the horizon. Of course, this presupposes that the initial state on $\mathcal{J}^-$ is nonsingular.

Next, we examine the outgoing null geodesics in the region exterior to the collapsing matter. These obey:

$$u \equiv t - 2\sqrt{2Mr} - r - 4M\log\left(\sqrt{r/2M} - 1\right) = \text{constant}. \tag{3.21}$$

Let us consider two geodesics, labelled by u_1 and u_2, which are separated in time by $\Delta u = u_2 - u_1$. On a constant t surface, Σ, the geodesics have a radial separation given by

$$\Delta u = r_1 - r_2 + 2(\sqrt{2Mr_1} - \sqrt{2Mr_2}) + 4M\log\left(\frac{\sqrt{r_1} - \sqrt{2M}}{\sqrt{r_2} - \sqrt{2m}}\right). \tag{3.22}$$

For large u_1, u_2, which corresponds to geodesics which reach $\mathcal{J}^+$ at late times, the radial separation of the geodesics near the horizon is determined by

$$\Delta u \approx 4M\log\left(\frac{\sqrt{r_1} - \sqrt{2M}}{\sqrt{r_2} - \sqrt{2M}}\right) \tag{3.23}$$

or

$$r_2 - 2M \approx (r_1 - 2M)e^{-\Delta u/4M}. \tag{3.24}$$

In other words, the outoging geodesics pile up along the horizon. Alternatively, we can note that all the geodesics which reach $\mathcal{J}^+$ after $u = \hat{u}$, where

$\hat{u} \gg M$, were contained on the surface $t = 0$ in the region between $r = 2M$ and $r = 2M + 4Me^{-\hat{u}/4M}$. This has two important consequences. First, we see that the late time radiation is entirely determined by the state of the field at distances arbitrarily close to the horizon. Second, an outgoing wave suffers an arbitrarily large redshift in escaping from near the horizon to $\mathcal{J}^+$. So, putting it all together, we see that to compute the late-time radiation we need only consider nonsingular states in a region near the horizon.

It is easy to construct a nonsingular state if we first define a new time coordinate. The trouble with the coordinate t used in (3.3) is that its flow is spacelike inside the horizon. However, this can be rectified if we define a new time, τ, as the value of t along the curve $dr + \sqrt{2M/r}dt = 0$. τ has the virtue of being nonsingular and timelike. We can choose a set of modes which are positive frequency with respect to τ, and expand the field in terms of them. Then, by the equivalence principle and standard quantum field theory in flat space, the corresponding vacuum will have a nonsingular stress-energy as seen by inertial observers. Of course, this choice of state is not unique since other nonsingular states can be obtained by applying particle creation operators to this state. This is irrelevant if one is only interested in late-time radiation, since the excitations above the vacuum will eventually redshift away.

Now at spatial infinity (more accurately: conformal infinity $\mathcal{I}_+$) the vacuum state is defined locally by the requirement that modes having positive frequency with respect to the variable $u = t_s - r_*$ are unoccupied, where t_s is Schwarzschild time and $r_* = r + 2M\ln(r - 2M)$ is the tortoise coordinate. We wish to find the relationship between this requirement and the preceding one.

The relationship between t and t_s is

$$t = t_s + 2\sqrt{2Mr} + 2M\ln\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}} \tag{3.25}$$

so that

$$u = t_s - r_* = t - 2\sqrt{2Mr} - r - 4M\ln(\sqrt{r} - \sqrt{2M})\ , \tag{3.26}$$

and thus one finds that along a curve with $dr + \sqrt{\frac{2M}{r}}dt = 0$,

$$\frac{du}{dt} = 2 + \sqrt{\frac{2M}{r}} + \frac{2M}{r - \sqrt{2Mr}}\ . \tag{3.27}$$

Because the last term on the right hand side is singular, the two definitions of positive frequency – with respect to u or to t – do not coincide. To remove the singularity, note that along any of the curves of interest $e^{-u/4M}$ has a simple zero at $r = 2M$, but is otherwise positive. Clearly then demanding positive frequency with respect to t along such curves requires positive frequency not with respect to u but rather with respect to

$$U = -e^{-u/4M} \ . \tag{3.28}$$

In this way we have arrived at the famous Unruh boundary conditions [26].

To summarize, the appropriate construction near the horizon is to expand the field as

$$\phi = \int \frac{d\omega}{\sqrt{2\pi 2\omega}} \left[b_\omega e^{i\omega U} + b_\omega^\dagger e^{-i\omega U} \right] \tag{3.29}$$

and take the state resulting from collapse to be $|0_U\rangle$, where $b_\omega |0_U\rangle = 0$. For simplicity, we have suppressed the field's dependence on the ingoing modes. It remains to describe this state in terms of particles defined by the modes $e^{-i\omega u}$. Since $u \to \infty$ at the horizon, we need another set of modes in which to expand the field inside the horizon. These will be referred to as $u_\omega^{in}(u)$; their explicit form will not be needed. We then have

$$\phi = \int \frac{d\omega}{\sqrt{2\pi 2\omega}} \left[\left(a_\omega e^{-i\omega u} + a_\omega^\dagger e^{i\omega u} \right) \Theta(-U) + \left(a_\omega^{in} u_\omega^{in} + a_\omega^{in\dagger} u_\omega^{in*} \right) \Theta(U) \right]. \tag{3.30}$$

The relation between the modes for $U < 0$ is given by

$$e^{i\omega' U(u)} = \int \frac{d\omega}{2\pi} \left[\alpha_{\omega'\omega} e^{-i\omega u} + \beta_{\omega'\omega} e^{i\omega u} \right] \quad ; \quad U < 0 \tag{3.31}$$

so

$$\alpha_{\omega'\omega} = \int_{-\infty}^{\infty} du \, e^{i(\omega' U(u) + \omega u)} \quad ; \quad \beta_{\omega'\omega} = \int_{-\infty}^{\infty} du \, e^{i(\omega' U(u) - \omega u)}, \tag{3.32}$$

or, since $U(u) = -e^{-u/4M}$,

$$\alpha_{\omega'\omega} = 4M \, \frac{\omega^{4\pi i\omega}}{\omega'} \, e^{2\pi M\omega} \, \Gamma(1 - 4iM\omega) \quad ; \quad \beta_{\omega'\omega} = 4M \, \frac{\omega^{-4\pi i\omega}}{\omega'} \, e^{-2\pi M\omega} \, \Gamma(1 + 4iM\omega). \tag{3.33}$$

So

$$|\alpha_{\omega'\omega}|^2 = \frac{(4M)^3\pi\omega}{\omega'^2}\frac{1}{1-e^{-8\pi M\omega}} \quad ; \quad |\beta_{\omega'\omega}|^2 = \frac{(4M)^3\pi\omega}{\omega'^2}\frac{e^{-8\pi M\omega}}{1-e^{-8\pi M\omega}}. \tag{3.34}$$

If we are only interested in the radiation which flows to infinity it is appropriate to form a density matrix by tracing over the states inside the horizon. To find the average number of particles radiated we should integrate $|\beta_{\omega'\omega}|^2$. But because the black hole radiates for an infinite amount of time at a constant rate (in the current approximation), this yields infinity. To find the *rate* of emission, we can place the hole in a large box and use the density of states $d\omega/2\pi$ for outgoing particles. For normalized modes the completeness relation, (3.18), and the relation

$$|\alpha_{\omega'\omega}|^2 = e^{8\pi M\omega}|\beta_{\omega'\omega}|^2 \tag{3.35}$$

imply

$$\int d\omega'|\beta_{\omega'\omega}|^2 = \frac{1}{e^{8\pi M\omega}-1}. \tag{3.36}$$

The rate of emission of particles in the range ω to $\omega + d\omega$ is then:

$$F(\omega) = \frac{d\omega}{2\pi}\frac{1}{e^{8\pi M\omega}-1}. \tag{3.37}$$

This is precisely the rate of emission from a black body, in one dimension, at temperature $T = 1/8\pi M$. F does not quite give the flux seen at infinity since some fraction, $1-\Gamma(\omega)$, of the particles will be reflected by the spacetime curvature back into the hole. Thus, for the flux at infinity we write,

$$F_\infty(\omega) = \frac{d\omega}{2\pi}\frac{\Gamma(\omega)}{e^{8\pi M\omega}-1}. \tag{3.38}$$

The preceding analysis shows that the average flux is that of a thermal body, not that the full density matrix is thermal. That the density matrix is exactly thermal is, in fact, easy to show by utilizing a clever trick due to Unruh [26]. We have not chosen to use this method here, and instead have gone through the rather laborious procedure of computing the Bogoliubov coefficients directly, because only the latter method can be used when one wants to find corrections to the spectrum.

3.3.1 Reissner-Nordstrom

A similar analysis can be carried out for the Reissner-Nordstrom geometry, the metric for which is, in the $L = 1$, $R = r$ gauge:

$$ds^2 = -dt^2 + (dr + \sqrt{2M/r - Q^2/r^2}\, dt)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (3.39)$$

The geometry has two horizons which are located at the two values of r for which t goes from being timelike to spacelike, and vice versa. The outer horizon radius is

$$R_+ = M + \sqrt{M^2 - Q^2} \qquad (3.40)$$

and the inner horizon radius is

$$R_- = M - \sqrt{M^2 - Q^2}. \qquad (3.41)$$

It should be noted that the coordinates in (3.39) fail to cover the region inside the inner horizon. This will not pose any obstacle to determining the radiance, though, as the calculation only depends in the geometry in the vicinity of the outer horizon.

In the present case, the state resulting from collapse is specified by demanding the absence of particles positive in frequency with respect to t along the curve $dr + \sqrt{2M/r - Q^2/r^2}$. Now the calculation can proceed in exact analogy to the uncharged case. The resulting temperature is:

$$T(M,Q) = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}. \qquad (3.42)$$

The temperature vanishes for the extremal black hole $M = Q$. If $Q > M$, consideration of the global geometry reveals that there is no black hole at all, but rather a naked singularity, which we will refer to as the meta-extremal case. Finally, if the radiated particles are themselves charged, the factor governing the emission probability is not the Boltzmann factor, $e^{-\omega/T}$, but rather:

$$\exp\left(-\frac{\omega - Qq/R_+(M,Q)}{T(M,Q)} \right) \qquad (3.43)$$

where the particles carry charge q. The factor Q/R_+ is the electrostatic potential at the outer horizon, and appears as a chemical potential causing particles with the same sign of charge as the hole to be preferentially emitted.

3.3.2 Breakdown of the Semiclassical Approximation

In the preceding treatment we have followed Hawking and considered the propagation of free fields on a classical background geometry, ignoring the back reaction altogether. It is usually assumed that the dominant effects of the back reaction can be incorporated by allowing the black hole to lose mass quasistatically, so that the radiation at infinity is thermal but with a slowly varying temperature. Indeed, this is how a normal thermal body is expected to behave during cooling. While this scenario seems quite plausible for a wide range of configurations, both for black holes and normal objects, there are notable instances where it is certain to be invalid, even qualitatively [27]. In particular, if the emission of a single typical quantum induces a relatively large change in temperature, a quasistatic description is clearly inappropriate. Since a typical quantum has energy equal to T, the temperature of the hole, we expect non-trivial back reaction effects to become important when

$$T\frac{\partial T}{\partial M} \approx T. \tag{3.44}$$

For a Schwarzschild black hole this happens when $M \approx m_p$, whereas for a Reissner-Nordstrom hole it happens when $M \approx Q$. In the Schwarzschild case, the breakdown occurs in a regime of high curvatures and large quantum fluctuations, presumably inaccessible to semiclassical methods. Not so, however, for a large near-extremal Reissner-Nordstrom hole, $M \approx Q \gg m_p$, for which the curvature near the horizon remains small, suggesting that meaningful calculations can be performed without the need for a full theory of quantum gravity. In the next chapter we perform such a calculation and see explicitly that the radiance from the near extremal hole is markedly different from what free field calculations indicate.

3.4 Effect of Fluctuations

The semiclassical approach to back reaction effects in quantum gravity is to compute $< T_{\mu\nu} >$ for a field quantized on a classical background geometry, and then to use this quantity on the right hand side of Einstein's equations,

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi < T_{\mu\nu} > . \tag{3.45}$$

The (generally intractable) problem then becomes to compute $< T_{\mu\nu} >$ for an arbitrary metric. One expects this approach to provide a reasonable description of the gross effects of back reaction provided that the fluctuations in $T_{\mu\nu}$ are not too large. For example, replacing $< T_{\mu\nu} >$ by $\sqrt{< {T_{\mu\nu}}^2 >}$ should not lead to any substantial change in the geometry if (3.45) is to be physically relevant.

In order to be quantitative, we shall consider the computation of stress energy fluctuations in the moving mirror model, which is discussed in [28, 29, 30]. We consider this model here because it provides a vivid demonstration of just how misleading the semiclassical approximation can be, and because we will referring to it later in another context. By "moving mirror model", we simply mean quantizing a scalar field in two dimensions subject to the condition that the field vanish along the timelike worldline $z_m(t)$. We will use the coordinates

$$u = t - z \quad ; \quad v = t + z. \tag{3.46}$$

Without the mirror present the propagator is

$$G(1,2) \equiv \langle 0|T[\phi(1)\phi(2)]|0\rangle = \frac{1}{4\pi}\log\left[(u_2 - u_1)(v_2 - v_1)\right] \tag{3.47}$$

and for a mirror at rest, $z_m(t) = 0$:

$$G(1,2) = \frac{1}{4\pi}\log\left[\frac{(u_2 - u_1)(v_2 - v_1)}{(u_2 - v_1)(v_2 - u_1)}\right]. \tag{3.48}$$

Now consider a general mirror trajectory, which we shall write in terms of u, v as $v_m(u)$. It is easiest to proceed by defining new coordinates in terms of which the mirror is at rest. Working in the region to the right of the mirror and defining

$$U(u) \equiv v_m(u) \quad ; \quad V(v) \equiv v, \tag{3.49}$$

we see that the trajectory $v = v_m(u)$ corresponds to $U = V$, which is the desired result. In the new coordinates the metric is

$$ds^2 = \left(U'(u)\right)^{-1} dU dV, \tag{3.50}$$

and because of conformal invariance, the propagator has the same form as before:

$$G(1,2) = \frac{1}{4\pi}\log\left[\frac{\left(U(u_2) - U(u_1)\right)(v_2 - v_1)}{\left(U(u_2) - v_1\right)(v_2 - v_1)}\right]. \tag{3.51}$$

The basic object we will be considering is the renormalized propagator defined by subtracting (3.47) from (3.51):

$$G_R(1,2) = \frac{1}{4\pi}\log\left[\frac{U(u_2)-U(u_1)}{(u_2-u_1)(U(u_2)-v_1)(v_2-U(u_1))}\right]. \tag{3.52}$$

This prescription simply corresponds to normal ordering with respect to standard Minkowski space creation and annihilation operators. $< T_{\mu\nu} >$ can now be found by operating on $G_R(1,2)$ according to

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\,\partial^\rho\phi, \tag{3.53}$$

so

$$< T_{\mu\nu} >= \lim_{1\leftrightarrow 2}\left[\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\rho}\frac{\partial}{\partial x_1^\rho}\frac{\partial}{\partial x_2^\alpha}\right]G_R(1,2). \tag{3.54}$$

In the present case the only nonzero element is

$$< T_{uu} >= \lim_{1\leftrightarrow 2}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}G_R(1,2) \tag{3.55}$$

which, after some algebra, is found to be

$$< T_{uu} >= \frac{1}{12\pi}\sqrt{v_m(u)}\frac{d^2}{du^2}\sqrt{\frac{1}{v_m(u)}}. \tag{3.56}$$

The result can be written in terms of the mirror trajectory $z_m(t)$ using

$$\frac{dv_m(u)}{du} = \frac{1+\dot{z_m}}{1-\dot{z_m}}, \tag{3.57}$$

yielding

$$< T_{uu} >= -\frac{1}{12\pi}\frac{(1-\dot{z_m}^2)^{\frac{1}{2}}}{(1-\dot{z_m})^2}\frac{d}{dt}\left[\frac{\ddot{z_m}}{(1-\dot{z_m}^2)^{\frac{3}{2}}}\right]. \tag{3.58}$$

These expressions hold in the region to the right of the mirror. The expressions to the left of the mirror are obtained by interchanging u and v.

These results can be used to find the radiation reaction force on the mirror. The force four-vector is, by energy conservation:

$$F_\mu = -\Delta T_{\mu\nu}v^\nu \tag{3.59}$$

where $\Delta T_{\mu\nu} = T_{\mu\nu}(+) - T_{\mu\nu}(-)$ is the difference in $T_{\mu\nu}$ evaluated on the right and left hand sides of the mirror, and v^ν is the mirror's velocity four-vector. We obtain:

$$F^\mu = \frac{1}{12\pi}\left(\frac{d^2v^\mu}{d\tau^2} - v^\mu v^\nu \frac{d^2 v_\nu}{d\tau^2}\right) \tag{3.60}$$

where τ is proper time. It is amusing to compare this to the radiation reaction force on a point charge in classical electrodynamics,

$$F^\mu = \frac{2e^2}{3}\left(\frac{d^2v^\mu}{d\tau^2} - v^\mu v^\nu \frac{d^2 v_\nu}{d\tau^2}\right). \tag{3.61}$$

This expression, containing as it does third derivatives of position of with respect to time, has some well known peculiar features such as runaway solutions and pre-acceleration. These carry over to the present case, although we hasten to add that a realistic mirror has a high frequency cutoff above which it becomes transparent; the worried reader need not be concerned with the hazards of runaway mirrors.

$< T_{\mu\nu}T_{\alpha\beta} >$ is found by operating on the renormalized four point function, which by Wick's theorem is:

$$G_R(1,2,3,4) = G_R(1,2)G_R(3.4) + G_R(1,3)G_R(2,4) + G_R(1,4)G_R(2,3). \tag{3.62}$$

We find, in the region to the right of the mirror:

$$< (T_{uu})^2 >= 3 < T_{uu} >^2 \qquad ; \qquad < (T_{vv})^2 >= 0$$

$$< T_{uu}T_{vv} >= -\frac{1}{8\pi^2}\frac{(dv_m/du)^2}{[v - v_m(u)]^4} \tag{3.63}$$

The fluctuations become infinite at the position of the mirror, $v = v_m(u)$. To see how this might affect the motion of the mirror we can consider the fluctuations in the radiation reaction force. Specifically, let us look at the force normal to the mirror,

$$F_N = -F_\mu n^\mu \tag{3.64}$$

where $n_\mu n^\mu = -1$ and $n_\mu v^\mu = 0$. The mean squared value is found to be:

$$< F_N^2 >= n^\mu v^\nu n^\alpha v^\beta [< T_{\mu\nu}(-)T_{\alpha\beta}(-) > + < T_{\mu\nu}(+)T_{\alpha\beta}(+) >$$

$$-2 < T_{\alpha\beta}(-) >< T_{\mu\nu}(+) >] \quad . \tag{3.65}$$

This is clearly infinite; for instance, the second term contributes

$$n^u n^v v^u v^v < T_{uu}(+) T_{vv}(+) >= \frac{1}{8\pi^2} \frac{(dv_m/du)^2}{[v - v_m(u)]^4}\Bigg|_{v=v_m(u)} . \tag{3.66}$$

The fluctuations diverge even for a stationary mirror. In the spirit of the semiclassical approach we might have attempted to model the back reaction by solving

$$ma_\mu = F_\mu . \tag{3.67}$$

However, in the case of the moving mirror this can not be interpreted as the leading term of a fully quantum mechanical treatment, since the fluctuations diverge. Instead, it seems more likely that a fully quantum mechanical treatment simply does not exist.

While these effects are most important for the moving mirror, there is no reason to believe that fluctuations are so violent in the case of a black hole much larger then the Planck mass. The point is simply that spacetime is locally flat near the horizon on a scale much larger than the Planck length. The divergences in the mirror model arose from the sharp boundary condition at the mirror, a feature which has no analog in the black hole case.

Chapter 4

Self-Interaction Corrections

Black hole radiance [3] was originally derived in an approximation where the background geometry was given, by calculating the response of quantum fields to this (collapse) geometry. As we have seen, in this approximation the radiation is thermal, and much has been made both of the supposed depth of this result and of the paradoxes that ensue if it is taken literally. For if the radiation is accurately thermal there is no connection between what went into the hole and what comes out, a possibility which is difficult to reconcile with unitary evolution in quantum theory – or, more simply, with the idea that there are equations uniquely connecting the past with the future. To address such questions convincingly, one must go beyond the approximation of treating the geometry as given, and treat it too as a quantum variable. This is not easy, and as far as we know no concrete correction to the original result has previously been derived in spite of much effort over more than twenty years. Here we shall calculate what is plausibly the leading correction to the emission rate of single particles in the limit of large Schwarzschild holes, by a method that can be generalized in several directions, as we shall outline.

There is a semi-trivial fact about the classic results for black hole radiation, that clearly prevents the radiation from being accurately thermal. This is the effect that the temperature of the hole depends upon its mass, so that in calculating the "thermal" emission rate one must know what mass of the black hole to use – but the mass is different, before and after the radiation! (Note that a rigorous identification of the temperature of a hot body from its radiation, can only be made for sufficiently high frequencies, such that the gray-body factors approach unity. But it is just in this limit that the ambiguity mentioned above is most serious.) As we have emphasized, this problem is particularly quantitatively acute for near-extremal holes — it is a general problem for bodies with finite heat capacity, and in the near-extremal limit the heat capacity of the black hole vanishes.

To resolve the above-mentioned ambiguity, one clearly must allow the geometry to fluctuate, namely to support black holes of different mass. Another point of view is that one must take into account the *self-gravitational interaction* of the radiation.

4.1 The Thin Shell Model

To obtain a complete description of a self-gravitating particle it would be necessary to compute the action for an arbitrary motion of the particle and gravitational field. While writing down a formal expression for such an object is straightforward, it is of little use in solving a concrete problem due to the large number of degrees of freedom present. To arrive at a more workable description of the particle-hole system, we will keep only those degrees of freedom which are most relevant to the problem of particle emission from regions of low curvature. The first important restriction is made by considering only spherically symmetric field configurations, and treating the particle as a spherical shell. This is an interesting case since black hole radiation into a scalar field occurs primarily in the s-wave, and virtual transitions to higher partial wave configurations are formally suppressed by powers of $\hbar$[1].

Before launching into the detailed calculation, which becomes rather intricate, it seems appropriate briefly to describe its underlying logic. After the truncation to s-wave, the remaining dynamics describes a shell of matter interacting with a black hole of fixed mass and with itself. (The mass as seen from infinity is the total mass, including that from the shell variable, and is allowed to vary. One could equally well have chosen the total mass constant, and allowed the hole mass to vary.) There is effectively one degree of freedom, corresponding to the position of the shell, but to isolate it one

[1] Since we do not address the ultraviolet problems of quantum gravity these corrections are actually infinite, but one might anticipate that in gravity theory with satisfactory ultraviolet behavior the virtual transitions will supply additive corrections of order $\frac{\omega^2\Lambda^2}{M^4}$, where Λ is the effective cutoff, and M is the mass of the black hole, but will not alter the exponential factors we compute.

must choose appropriate variables and solve constraints, since the original action superficially appears to contain much more than this. Having done that, one obtains an effective action for the true degree of freedom. This effective action is nonlocal, and its full quantization would require one to resolve factor-ordering ambiguities, which appears very difficult. Hence we quantize it semi-classically, essentially by using the WKB approximation. After doing this one arrives at a non-linear first order partial differential equation for the phase of the wave function. This differential equation may be solved by the method of characteristics. According to this method, one solves for the characteristics, specifies the function to be determined along a generic initial surface (intersecting the characteristics transversally), and evolves the function away from the initial surface, by integrating the action along the characteristics. (For a nice brief account of this, see [31].)

When the background geometry is regarded as fixed the characteristics for particle motion are simply the geodesics in that geometry, and they are essentially independent of the particle's mass or energy — principle of equivalence — except that null geodesics are used for massless particles, and timelike geodesics for massive particles. Here we find that the characteristics depend on the mass and energy in a highly non-trivial way. Also the action along the characteristics, which would be zero for a massless particle and proportional to the length for a massive particle, is now a much more complicated expression. Nevertheless we can solve the equations, to obtain the proper modes for our problem.

Having obtained the modes, the final step is to identify the state of the quantum field — that is, the occupation of the modes — appropriate to the physical conditions we wish to describe. We do this by demanding that a freely falling observer passing through the horizon see no singular behavior, and that positive frequency modes are unoccupied in the distant past. This, it has been argued, is plausibly the appropriate prescription for the state of the quantum field excited by collapse of matter into a black hole, at least in so far as it leads to late-time radiation. Using it, we obtain a mixture of positive- and negative- frequency modes at late times, which can be interpreted as a state of radiation from the hole. For massless scalar particles, we carry the explicit calculation far enough to identify the leading correction to the exponential dependence of the radiation intensity on frequency.

4.1.1 Effective Action

We now derive the Hamiltonian effective action for a self-gravitating particle in the s-wave. First, we would like to explain why the Hamiltonian form of the action is particularly well suited to our problem. As explained above, our physical problem really contains just one degree of freedom, but the original action appears to contain several. The reason of course is that Einstein gravity is a theory with constraints and one should only include a subset of the spherically symmetric configurations in the physical description, namely those satisfying the constraints. In general, in eliminating constraints Hamiltonian methods are more flexible than Lagrangian methods. This appears to be very much the case for our problem, as we now discuss.

In terms of the variables appearing in the Lagrangian description, the constraints have the form

$$\mathcal{C}_L\left[\hat{r}, \dot{\hat{r}}; g_{\mu\nu}, \dot{g_{\mu\nu}}\right] = 0,$$

where $\hat{r}$ is the shell radius, and $\dot{}$ represents $\frac{d}{dt}$. When applied to the spherically symmetric, source free, solutions, one obtains the content of Birkhoff's theorem – the unique solution is the Schwarzschild geometry with some mass, M. Since this must hold for the regions interior and exterior to the shell (with a different mass M for each), and since M must be time independent, we see that only those shell trajectories which are "energy conserving" are compatible with the constraints. This feature makes the transition to the quantum theory rather difficult, as one desires an expression for the action valid for an arbitrary shell trajectory. This defect is remedied in the Hamiltonian formulation, where the constraints are expressed in terms of momenta rather than time derivatives,

$$\mathcal{C}_H\left[\hat{r}, p; g_{ij}, \pi_{ij}\right] = 0.$$

At each time, the unique solution is again some slice of the Schwarzschild geometry, but the constraints no longer prevent M from being time dependent. Thus, an arbitrary shell trajectory $\hat{r}(t)$, $p(t)$, is perfectly consistent with the Hamiltonian form of the constraints, making quantization much more convenient.

As before, we begin by writing the metric in ADM form:

$$ds^2 = -N^t(t,r)^2dt^2 + L(t,r)^2[dr + N^r(t,r)dt]^2 + R(t,r)^2[d\theta^2 + \sin\theta^2 d\phi^2] \quad (4.1)$$

In considering the above form, we have restricted ourselves to spherically symmetric geometries at the outset. With this choice of variables, the action for the shell is written

$$S^s = -m \int \sqrt{-\hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu} = -m \int dt \sqrt{\hat{N}^{t2} - \hat{L}^2 \left(\dot{\hat{r}} + \hat{N}^r\right)^2}, \tag{4.2}$$

m representing the rest mass of the shell, and the carets instructing one to evaluate quantities at the shell $\left(\hat{g}_{\mu\nu} = g_{\mu\nu}(\hat{t}, \hat{r})\right)$.

The action for the gravity-shell system is then

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}\,\mathcal{R} - m \int dt \sqrt{(\hat{N}^t)^2 - \hat{L}^2(\dot{\hat{r}} + \hat{N}^r)^2} + \text{ boundary terms} \tag{4.3}$$

and can be written in canonical form as

$$S = \int dt\, p\,\dot{\hat{r}} + \int dt\, dr\, [\pi_R \dot{R} + \pi_L \dot{L} - N^t(\mathcal{H}_t^s + \mathcal{H}_t^G) - N^r(\mathcal{H}_r^s + \mathcal{H}_r^G)] - \int dt\, M_{\text{ADM}} \tag{4.4}$$

with

$$\mathcal{H}_t^s = \sqrt{(p/\hat{L})^2 + m^2}\ \delta(r - \hat{r}) \quad ; \quad \mathcal{H}_r^s = -p\,\delta(r - \hat{r}) \tag{4.5}$$

$$\mathcal{H}_t^G = \frac{L{\pi_L}^2}{2R^2} - \frac{\pi_L \pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} \quad ; \quad \mathcal{H}_r^G = R'\pi_R - L\pi_L' \tag{4.6}$$

where $'$ represents $\frac{d}{dr}$. M_{ADM} is the ADM mass of the system, and is numerically equal to the total mass of the combined gravity-shell system.

We now wish to eliminate the gravitational degrees of freedom in order to obtain an effective action which depends only on the shell variables. To accomplish this, we first identify the constraints which are obtained by varying with respect to N^t and N^r:

$$\mathcal{H}_t = \mathcal{H}_t^s + \mathcal{H}_t^G = 0 \quad ; \quad \mathcal{H}_r = \mathcal{H}_r^s + \mathcal{H}_r^G = 0. \tag{4.7}$$

By solving these constraints, and inserting the solutions back into (4.4) we can eliminate the dependence on π_R and π_L. We first consider the linear combination of constraints

$$0 = \frac{R'}{L}\mathcal{H}_t + \frac{\pi_L}{RL}\mathcal{H}_r = -\mathcal{M}' + \frac{\hat{R}'}{\hat{L}}\mathcal{H}_t^s + \frac{\hat{\pi}_L}{\hat{R}\hat{L}}\mathcal{H}_r^s \tag{4.8}$$

where

$$\mathcal{M} = \frac{{\pi_L}^2}{2R} + \frac{R}{2} - \frac{RR'^2}{2L^2}. \tag{4.9}$$

Away from the shell the solution of this constraint is simply $\mathcal{M}$ = constant. By considering a static slice ($\pi_L = \pi_R = 0$), we see that the solution is a static slice of the Schwarzschild geometry with $\mathcal{M}$ the corresponding mass parameter. The presence of the shell causes $\mathcal{M}$ to be discontinuous at $\hat{r}$, so we write

$$\begin{aligned} \mathcal{M} &= M \qquad r < \hat{r} \\ \mathcal{M} &= M_+ \qquad r > \hat{r}. \end{aligned} \tag{4.10}$$

As there is no matter outside the shell we also have $M_{\mathrm{ADM}} = M_+$. Then, using (4.8) and (4.9) we can solve the constraints to find π_L and π_R:

$$\begin{aligned} \pi_L &= R\sqrt{(R'/L)^2 - 1 + 2M/R} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L' \qquad r < \hat{r} \\ \pi_L &= R\sqrt{(R'/L)^2 - 1 + 2M_+/R} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L'; \qquad r > \hat{r}. \end{aligned} \tag{4.11}$$

The relation between M_+ and M is found by solving the constraints at the position of the shell. This is done most easily by choosing coordinates such that L and R are continuous as one crosses the shell, and $\pi_{R,L}$ are free of singularities there. Then, integration of the constraints across the shell yields

$$\begin{aligned} \pi_L(\hat{r}+\epsilon) - \pi_L(\hat{r}-\epsilon) &= -p/\hat{L} \\ R'(\hat{r}+\epsilon) - R'(\hat{r}-\epsilon) &= -\frac{1}{\hat{R}}\sqrt{p^2 + m^2\hat{L}^2} \end{aligned} \tag{4.12}$$

Now, when the constraints are satisfied a variation of the action takes the form

$$dS = p\, d\hat{r} + \int dr(\pi_R \delta R + \pi_L \delta L) - M_+\, dt \tag{4.13}$$

where $\pi_{R,L}$ are now understood to be given by (4.11), and M_+ is determined by solving (4.12). We wish to integrate the expression (4.13) to find the action for an arbitrary shell trajectory. As discussed above, the geometry inside the shell is taken to be fixed (namely, M is held constant) while the geometry outside the shell will vary in order to satisfy the constraints. It is easiest to integrate the action by initially varying the geometry away from the shell. We first consider starting from an arbitrary geometry and varying L until $\pi_R = \pi_L = 0$, while holding $\hat{r}, p, R, \hat{L}$ fixed:

$$\begin{aligned}
\int dS &= \int_{r_{\min}}^{\infty} dr \int_{\pi=0}^{L} \delta L \, \pi_L \\
&= \int_{r_{\min}}^{\hat{r}-\epsilon} dr \int_{\pi=0}^{L} \delta L \, R\sqrt{(R'/L)^2 - 1 + 2M/R} \, + \int_{\hat{r}+\epsilon}^{\infty} dr \int_{\pi=0}^{L} \delta L \, R\sqrt{(R'/L)^2 - 1 + 2M_+/R} \\
&= \int_{r_{\min}}^{\hat{r}-\epsilon} dr \left[RL\sqrt{(R'/L)^2 - 1 + 2M/R} \, + RR' \log \left| \frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M/R}}{\sqrt{|1-2M/R|}} \right| \right] \\
&\quad + \int_{\hat{r}+\epsilon}^{\infty} dr \left[RL\sqrt{(R'/L)^2 - 1 + 2M_+/R} \, + RR' \log \left| \frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M_+/R}}{\sqrt{|1-2M_+/R|}} \right| \right]
\end{aligned} \tag{4.14}$$

where the lower limit of integration, r_{min}, properly extends to the collapsing matter forming the black hole; its precise value will not be important. We have discarded the constant arising from the lower limit of the L integration. In the next stage we can vary L and R,while keeping $\pi_{R,L} = 0$, to some set geometry. Since the momenta vanish, there is no contribution to the action from this variation.

It remains to consider nonzero variations at the shell. If an arbitrary variation of L and R is inserted into the final expression of (4.14) one finds

$$dS = \int_{r_{\min}}^{\infty} dr \, [\pi_R \delta R + \pi_L \delta L] - \left[\frac{\partial S}{\partial \hat{R}'}(\hat{r} + \epsilon) - \frac{\partial S}{\partial \hat{R}'}(\hat{r} - \epsilon) \right] d\hat{R} + \frac{\partial S}{\partial M_+} dM_+ . \tag{4.15}$$

Since R' is discontinuous at the shell,

$$\frac{\partial S}{\partial \hat{R}'}(\hat{r} + \epsilon) - \frac{\partial S}{\partial \hat{R}'}(\hat{r} - \epsilon)$$

is nonvanishing and needs to be subtracted in order that the relations

$$\frac{\delta S}{\delta R} = \pi_R \quad ; \quad \frac{\delta S}{\delta L} = \pi_L$$

will hold. From (4.14), the term to be subtracted is

$$-\left[\frac{\partial S}{\partial \hat{R}'}(\hat{r}+\epsilon)-\frac{\partial S}{\partial \hat{R}'}(\hat{r}-\epsilon)\right] d\hat{R}$$

$$= -d\hat{R}\hat{R}\log\left|\frac{R'(\hat{r}-\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2-1+2M/\hat{R}}}{\sqrt{|1-2M/\hat{R}|}}\right|$$

$$+ d\hat{R}\hat{R}\log\left|\frac{R'(\hat{r}+\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}+\epsilon)/\hat{L})^2-1+2M/\hat{R}}}{\sqrt{|1-2M_+/\hat{R}|}}\right|. \qquad (4.16)$$

Similarly, arbitrary variations of L and R induce a variation of M_+ causing the appearance of the final term in (4.15). Thus we need to subtract

$$\frac{\partial S}{\partial M_+}dM_+ = -\int_{\hat{r}+\epsilon}^{\infty} dr\, L\frac{\sqrt{(R'/L)^2-1+2M_+/R}}{1-2M_+/R}dM_+. \qquad (4.17)$$

Finally, we consider variations in $p, \hat{r}$, and t. t variations simply give $dS = -M_+dt$. We do not need to separately consider variations of p and $\hat{r}$, since when the constraints are satisfied their variations are already accounted for in our expression for S, as will be shown.

Collecting all of these terms, our final expression for the action reads

$$S = \int_{r_{\min}}^{\hat{r}-\epsilon} dr\, \left[RL\sqrt{(R'/L)^2-1+2M/R} + RR'\log\left|\frac{R'/L-\sqrt{(R'/L)^2-1+2M/R}}{\sqrt{|1-2M/R|}}\right|\right]$$

$$+\int_{\hat{r}+\epsilon}^{\infty} dr\, \left[RL\sqrt{(R'/L)^2-1+2M_+/R} + RR'\log\left|\frac{R'/L-\sqrt{(R'/L)^2-1+2M_+/R}}{\sqrt{|1-2M_+/R|}}\right|\right]$$

$$-\int dt\, \dot{\hat{R}}\hat{R}\left[\log\left|\frac{R'(\hat{r}-\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2-1+2M/\hat{R}}}{\sqrt{|1-2M/\hat{R}|}}\right|\right.$$

$$\left.+\log\left|\frac{R'(\hat{r}+\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}+\epsilon)/\hat{L})^2-1+2M/\hat{R}}}{\sqrt{|1-2M_+/\hat{R}|}}\right|\right]$$

$$+\int dt \int_{\hat{r}+\epsilon}^{\infty} dr \, \frac{L\sqrt{(R'/L)^2 - 1 + 2M_+/R}}{1 - 2M_+/R} \dot{M}_+ - \int dt \, M_+. \tag{4.18}$$

To show that this is the correct expression we can differentiate it; then it can be seen explicitly that when the constraints are satisfied (4.13) holds.

We now wish to write the action in a more conventional form as the time integral of a Lagrangian. As it stands, the action in (4.18) is given for an arbitrary choice of L and R consistent with the constraints. There is, of course, an enormous amount of redundant information contained in this description, since many L's and R's are equivalent to each other through a change of coordinates. To obtain an action which only depends on the truly physical variables $p, \hat{r}$ we make a specific choice for L and R, *ie.* choose a gauge. In so doing, we must respect the condition

$$R'(\hat{r}+\epsilon) - R'(\hat{r}-\epsilon) = -\frac{1}{\hat{R}}\sqrt{p^2 + m^2\hat{L}^2}$$

which constrains the form of R' arbitrarily near the shell. Suppose we choose R for all $r > \hat{r}$; then $R'(\hat{r}-\epsilon)$ is fixed by the constraint, but we can still choose R for $r < \hat{r}-\epsilon$, in other words, away from the shell. We will let $R'_<$ denote the value of R' close to the shell but far enough away such that R is still freely specifiable. We employ the analogous definition for $R'_>$, except in this case we are free to choose $R'_> = R'(\hat{r}+\epsilon)$.

In terms of this notation the time derivative of S is

$$L = \frac{dS}{dt} = \dot{\hat{r}}\hat{R}\hat{L}\left[\sqrt{(R'_</\hat{L})^2 - 1 + 2M/\hat{R}} - \sqrt{(R'_>/\hat{L})^2 - 1 + 2M_+/\hat{R}}\right]$$

$$-\dot{\hat{R}}\hat{R}\log\left|\frac{R'(\hat{r}-\epsilon)/\hat{L} - \sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2 - 1 + 2M/\hat{R}}}{R'_</\hat{L} - \sqrt{(R'_</\hat{L})^2 - 1 + 2M/\hat{R}}}\right|$$

$$+\int_{r_{\min}}^{\hat{r}-\epsilon} dr[\pi_R\dot{R} + \pi_L\dot{L}] + \int_{\hat{r}+\epsilon}^{\infty} dr[\pi_R\dot{R} + \pi_L\dot{L}] - M_+. \tag{4.19}$$

At this point we will, for simplicity, specialize to a massless particle ($m = 0$) and define $\eta = \pm = \mathrm{sgn}(p)$. Then the constraints (4.12) read

$$R'(\hat{r}-\epsilon) = R'(\hat{r}+\epsilon) + \frac{\eta p}{\hat{R}}$$

$$\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2 - 1 + 2M/\hat{R}} = \sqrt{(R'(\hat{r}+\epsilon)/\hat{L})^2 - 1 + 2M_+/\hat{R}} + \frac{p}{\hat{L}\hat{R}}. \tag{4.20}$$

These relations can be inserted into (4.19) to yield

$$\begin{aligned} L = \dot{\hat{r}}\hat{R}\hat{L}&\left[\sqrt{(R'_</\hat{L})^2 - 1 + 2M/\hat{R}} - \sqrt{(R'_>/\hat{L})^2 - 1 + 2M_+/\hat{R}}\right] \\ &-\eta\dot{\hat{R}}\hat{R}\log\left|\frac{R'_>/\hat{L} - \eta\sqrt{(R'_>/\hat{L})^2 - 1 + 2M_+/\hat{R}}}{R'_</\hat{L} - \eta\sqrt{(R'_</\hat{L})^2 - 1 + 2M/\hat{R}}}\right| \\ &+\int_{r_{\min}}^{\hat{r}-\epsilon} dr[\pi_R\dot{R} + \pi_L\dot{L}] + \int_{\hat{r}+\epsilon}^{\infty} dr[\pi_R\dot{R} + \pi_L\dot{L}] - M_+. \end{aligned} \tag{4.21}$$

Now we can use the freedom to choose a gauge to make (4.21) appear as simple as possible. It is clearly advantageous to choose L and R to be time independent, so $\pi_R\dot{R} + \pi_L\dot{L} = 0$. Also, having $R' = L$ simplifies the expressions further. Finally, it is crucial that the metric be free of coordinate singularities. A gauge which conveniently accommodates these features is

$$L = 1 \quad ; \quad R = r$$

The $L = 1, R = r$ gauge reduces the Lagrangian to

$$L = \dot{\hat{r}}[\sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}}] - \eta\dot{\hat{r}}\hat{r}\log\left|\frac{\sqrt{\hat{r}} - \eta\sqrt{M_+}}{\sqrt{\hat{r}} - \eta\sqrt{2M}}\right| - M_+ \tag{4.22}$$

where M_+ is now found from the constraints (4.20) to be related to p by

$$p = \frac{M_+ - M}{\eta - \sqrt{2M_+/r}}. \tag{4.23}$$

The canonical momentum conjugate to $\hat{r}$ obtained from 4.22 is

$$p_c = \frac{\partial L}{\partial\dot{\hat{r}}} = \sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}} - \eta\hat{r}\log\left|\frac{\sqrt{\hat{r}} - \eta\sqrt{2M_+}}{\sqrt{\hat{r}} - \eta\sqrt{2M}}\right| \tag{4.24}$$

in terms of which we write the action in canonical form as

$$S = \int dt[p_c\dot{\hat{r}} - M_+] \tag{4.25}$$

which identifies M_+ as the Hamiltonian. We should point out that M_+ is the Hamiltonian only for a restricted set of gauges. If we look back at (4.21) we see that the terms $\pi_R\dot{R} + \pi_L\dot{L}$ will in general contribute to the Hamiltonian.

4.1.2 Quantization

In this section we discuss the quantization of the effective action (4.25). First, it is convenient to rewrite the action in a form which explicitly separates out the contribution from the particle. We write

$$M_+ = M - p_t$$

so

$$S = \int dt[p_c\dot{r} + p_t] \tag{4.26}$$

and the same substitution is understood to be made in (4.24). We have omitted a term, $\int dtM$, which simply contributes an overall constant to our formulas. In order to place our results in perspective, it is useful to step back and consider the analogous expressions in flat space. Our results are an extension of

$$p = \pm\sqrt{p_t{}^2 - m^2} \tag{4.27}$$

$$S = \int dt[p\dot{r} + p_t]. \tag{4.28}$$

Indeed, the $G \to 0$ limit of (4.24), (4.25) yields precisely these expressions (with
$m = 0$). To quantize, one is tempted to insert the substitutions $p \to -i\frac{\partial}{\partial r}$, $p_t \to -i\frac{\partial}{\partial t}$ into (4.27), so as to satisfy the canonical commutation relations. This results in a rather unwieldy, nonlocal differential equation. In this trivial case we know, of course, that the correct description of the particle is obtained by demanding locality and squaring both sides of (4.27) before substituting p and p_t. So for this example it is straightforward to move from the point particle description to the field theory description, *i.e.* the Klein-Gordon equation. Now, returning to (4.24) we are again met with the question of how to implement the substitutions $p \to -i\partial$. In this case the difficulty is more severe; we no longer have locality as a guiding criterion instructing us how to manipulate (4.24) before turning the p's into differential operators. This is because we expect the effective action (4.26) to be nonlocal on physical grounds, as it was obtained by including the gravitational field of the shell.

There is, however, a class of solutions to the field equations for which this ambiguity is irrelevant to leading order, and which is sufficient to determine the late-time radiation from a black hole. These are the short-wavelength solutions, which are accurately described by the geometrical optics, or WKB, approximation. Writing these solutions as

$$\phi(t,r) = e^{iS(t,r)},$$

the condition determining the validity of the WKB approximation is that

$$|\partial S| \gg |\partial^2 S|^{1/2},\ |\partial^3 S|^{1/3}\ \dots$$

and that the geometry is slowly varying compared to S. In this regime, derivatives acting on $\phi(t,r)$ simply bring down powers of ∂S, so we can make the replacements

$$p_c \to \frac{\partial S}{\partial r} \quad ; \quad p_t \to \frac{\partial S}{\partial t}$$

and obtain a Hamilton-Jacobi equation for S. Furthermore, it is well known that the solution of the Hamilton-Jacobi equation is just the classical action. So, if $\hat{r}(t)$ is a solution of the equations of motion found by extremizing (4.26), then

$$S\left(t,\hat{r}(t)\right) = S\left(0,\hat{r}(0)\right) + \int_0^t dt\left[p_c\left(\hat{r}(t)\right)\dot{\hat{r}}(t) + p_t\right] \tag{4.29}$$

where

$$p_c\left(0,\hat{r}\right) = \frac{\partial S}{\partial r}\left(0,\hat{r}\right). \tag{4.30}$$

Since the Lagrangian in (4.26) has no explicit time dependence, the Hamiltonian p_t is conserved. Using this fact, it is easy to verify that the trajectories, $\hat{r}(t)$, which extremize (4.26) are simply the null geodesics of the metric

$$ds^2 = -dt^2 + \left(dr + \sqrt{\frac{2M_+}{r}}dt\right)^2. \tag{4.31}$$

From (3.8) the geodesics are:

$$\begin{aligned}\text{ingoing:}\quad & t + \hat{r}(t) + 2\sqrt{2M_+\hat{r}(t)} + 4M_+\log\left[\sqrt{\hat{r}(t)} + \sqrt{2M_+}\right] \\ & = \hat{r}(0) + 2\sqrt{2M_+\hat{r}(0)} + 4M_+\log\left[\sqrt{\hat{r}(0)} + \sqrt{2M_+}\right]\end{aligned}$$

$$\text{outgoing:} \quad t - \hat{r}(t) - 2\sqrt{2M_+\hat{r}(t)} - 4M_+ \log\,[\sqrt{\hat{r}(t)} - \sqrt{2M_+}]$$
$$= -\hat{r}(0) - 2\sqrt{2M_+\hat{r}(0)} - 4M_+ \log\,[\sqrt{\hat{r}(0)} - \sqrt{2M_+}]. \quad (4.32)$$

M_+, in turn, is determined by the initial condition $S(0,r)$ according to (4.24) and (4.30):

$$\text{ingoing:} \quad \frac{\partial S}{\partial r}(0,\hat{r}(0)) = \sqrt{2M\hat{r}(0)} - \sqrt{2M_+\hat{r}(0)} + \hat{r}(0)\log\left|\frac{\sqrt{\hat{r}(0)}+\sqrt{2M_+}}{\sqrt{\hat{r}(0)}+\sqrt{2M}}\right|$$

$$\text{outgoing:} \quad \frac{\partial S}{\partial r}(0,\hat{r}(0)) = \sqrt{2M\hat{r}(0)} - \sqrt{2M_+\hat{r}(0)} - \hat{r}(0)\log\left|\frac{\sqrt{\hat{r}(0)}-\sqrt{2M_+}}{\sqrt{\hat{r}(0)}-\sqrt{2M}}\right|. \quad (4.33)$$

Finally, we can use this value of M_+ to determine $p_c(t)$:

$$\text{ingoing:} \quad p_c(t) = \sqrt{2M\hat{r}(t)} - \sqrt{2M_+\hat{r}(t)} + \hat{r}(t)\log\left|\frac{\sqrt{\hat{r}(t)}+\sqrt{2M_+}}{\sqrt{\hat{r}(t)}+\sqrt{2M}}\right|$$

$$\text{outgoing:} \quad p_c(t) = \sqrt{2M\hat{r}(t)} - \sqrt{2M_+\hat{r}(t)} - \hat{r}(t)\log\left|\frac{\sqrt{\hat{r}(t)}-\sqrt{2M_+}}{\sqrt{\hat{r}(t)}-\sqrt{2M}}\right|. \quad (4.34)$$

These formulas are sufficient to compute $S(t,r)$ given $S(0,r)$.

As will be discussed in the next section, the relevant solutions needed to describe the state of the field following black hole formation are those with the initial condition

$$S(0,r) = kr \qquad k > 0 \quad (4.35)$$

near the horizon. Here, k must be large ($\gg 1/M$) if the solution is to be accurately described by the WKB approximation. In fact, the relevant k's needed to calculate the radiation from the hole at late times become arbitrarily large, due to the ever increasing redshift experienced by the emitted quanta as they escape to infinity. We also show in the next section that to compute the emission probability of a quantum of frequency ω, we are required to find the solution for all times in the region between $r = 2M$ and $r = 2(M + \omega)$. That said, we turn to the calculation of $S(t,r)$ in this region, and with the initial condition (4.35). The solutions are determined from (4.29), (4.32)-(4.34). Because of the large redshift, we only need to keep those terms in these relations which become singular near the horizon. We then have for the outgoing solutions:

$$S(t,r) = k\,\hat{r}(0) - \int_{\hat{r}(0)}^{r} d\hat{r}\,\hat{r}\log\left[\frac{\sqrt{\hat{r}}-\sqrt{2M_+}}{\sqrt{\hat{r}}-\sqrt{2M}}\right] - (M_+ - M)t \tag{4.36}$$

$$t - 4M_+\log\left[\sqrt{r}-\sqrt{2M_+}\right] = -4M_+\log\left[\sqrt{\hat{r}(0)}-\sqrt{2M_+}\right] \tag{4.37}$$

$$k = -\hat{r}(0)\log\left[\frac{\sqrt{\hat{r}(0)}-\sqrt{2M_+}}{\sqrt{\hat{r}(0)}-\sqrt{2M}}\right]. \tag{4.38}$$

To complete the calculation, we need to invert (4.37) and (4.38) to find M_+ and $\hat{r}(0)$ in terms of t and r, and then insert these expressions into (4.36). One finds that to next to leading order,

$$\sqrt{2M_+} = \sqrt{2M} + (\sqrt{r}-\sqrt{2M})\frac{(e^{k/2M'}-1)e^{-t/4M'}}{1+(e^{k/2M'}-1)e^{-t/4M'}}$$

$$\sqrt{\hat{r}(0)} = \sqrt{2M} + (\sqrt{r}-\sqrt{2M})\frac{e^{(k/2M'-t/4M')}}{1+(e^{k/2M'}-1)e^{-t/4M'}} \tag{4.39}$$

where

$$M' = M + \sqrt{2M}(\sqrt{r}-\sqrt{2M})\frac{e^{(k/2M-t/4M)}}{1+e^{(k/2M-t/4M)}}. \tag{4.40}$$

Plugging these relations into (4.36) and keeping only those terms which contribute to the late-time radiation, one finds after some tedious algebra,

$$S(t,r) = -(2M^2 - r^2/2)\log\left[1+e^{(k/2M'-t/4M')}\right]. \tag{4.41}$$

4.1.3 Results

We will now discuss the application of these results to the problem of black hole radiance. The procedure is a slight modification of the one we discussed in the free field theory case. As before, the point is that there are two inequivalent sets of modes which need to be considered: those which are natural from the standpoint of an observer making measurements far from the black hole, and those which are natural from the standpoint of an observer freely falling through the horizon subsequent to the collapse of the infalling matter. The appropriate modes for the observer at infinity are those which are positive frequency with respect to the Killing time, t. Writing these modes as

$$u_k(r)e^{-i\omega_k t},$$

$\hat{\phi}(t,r)$ reads

$$\hat{\phi}(t,r) = \int dk \left[\hat{a}_k u_k(r) e^{-i\omega_k t} + \hat{a}_k^\dagger u_k^*(r) e^{i\omega_k t}\right]. \tag{4.42}$$

These modes are singular at the horizon,

$$\frac{du_k}{dr} \to \infty \quad \text{as} \quad r \to 2M.$$

Symptomatic of this is that the freely falling observer would measure an infinite energy-momentum density in the corresponding vacuum state,

$$\langle 0_t | T_{\mu\nu} | 0_t \rangle \to \infty \quad \text{as} \quad r \to 2M$$

where $\hat{a}_k|0_t\rangle = 0$. However, we do not expect this to be the state resulting from collapse, since the freely falling observer is not expected to encounter any pathologies in crossing the horizon, where the local geometry is entirely nonsingular for a large black hole. To describe the state resulting from collapse, it is more appropriate to use modes which extend smoothly through the horizon, and which are positive frequency with respect to the freely falling observer. Denoting a complete set of such modes by $v_k(t,r)$, we write

$$\hat{\phi}(t,r) \int dk \left[\hat{b}_k v_k(t,r) + \hat{b}_k^\dagger v_k^*(t,r)\right]. \tag{4.43}$$

Then, the state determined by

$$\hat{b}_k |0_v\rangle = 0$$

results in a non-singular energy-momentum density at the horizon, and so is a viable candidate. The operators $\hat{a}_k$ and $\hat{b}_k$ are related by the Bogoliubov coefficients,

$$\hat{a}_k = \int dk' \left[\alpha_{kk'} \hat{b}_{k'} + \beta_{kk'} \hat{b}_{k'}^\dagger\right] \tag{4.44}$$

where

$$\alpha_{kk'} = \frac{1}{2\pi u_k(r)} \int_{-\infty}^{\infty} dt\, e^{i\omega_k t} v_{k'}(t,r)$$

$$\beta_{kk'} = \frac{1}{2\pi u_k(r)} \int_{-\infty}^{\infty} dt\, e^{i\omega_k t} v^*_{k'}(t,r). \tag{4.45}$$

Note that here we compute the Bogoliubov coefficients by performing a t integration, rather than an integration over a spatial coordinate, as is conventional. We are forced to do this in the present case since we do not know the spatial dependence of the definite energy modes. The flux seen at infinity is

$$F_\infty(\omega_k) = \frac{d\omega_k}{2\pi} \frac{\Gamma(\omega_k)}{|\alpha_{kk'}/\beta_{kk'}|^2 - 1}. \tag{4.46}$$

Next, we consider the issue of determining the modes $v_k(t,r)$. As stated above, we require these modes to be nonsingular at the horizon. Since the metric near the horizon is a smooth function of t and r, a set of such modes can be defined by taking their behaviour on a constant time surface, say $t = 0$, to be

$$v_k(0,r) \approx e^{ikr} \quad \text{as} \quad r \to 2M.$$

This is, of course, the initial condition given in (4.35). Now, the integrals in (4.45) determining the Bogoliubov coefficients depend on the values of $v_k(t,r)$ at constant r. Since v_k is evaluated in the WKB approximation, the highest accuracy will be obtained when r is as close to the horizon as possible, since that is where v_k's wavelength is short. On the other hand, in calculating the emission of a particle of energy ω_k, we cannot take r to be less than $2(M+\omega_k)$, since the solution $u_k(r)e^{-i\omega_k t}$ cannot be extended past that point. Therefore, we calculate the integrals with $r = 2(M+\omega_k)$.

The results of the previous section give us an explicit expression for v_k. From (4.41),

$$v_k(t, 2(M+\omega_k)) = e^{iS(t,2(M+\omega_k))} = e^{i(4M\omega_k + 2\omega^2)\log[1+e^{(k/2M'-t/4M')}]} \tag{4.47}$$

where M' is

$$M' = M + \sqrt{2M}(\sqrt{2(M+\omega_k)} - \sqrt{2M}) \frac{e^{(k/2M-t/4M)}}{1+e^{(k/2M-t/4M)}} \approx M + \omega_k \frac{e^{(k/2M-t/4M)}}{1+e^{(k/2M-t/4M)}}. \tag{4.48}$$

Then, the integrals are,

$$\int_{-\infty}^{\infty} dt e^{i\omega_k t} e^{\pm i(4M\omega_{k'}+2\omega_{k'}^2)\log\left[1+e^{(k'/2M'-t/4M')}\right]}, \tag{4.49}$$

the upper sign corresponding to $\alpha_{kk'}$, and the lower to $\beta_{kk'}$. We can compute the integrals using the saddle point approximation. It is readily seen that for the upper sign, the saddle point is reached when

$$e^{(k'/2M'-t/4M')} \to \infty,$$

so t is on the real axis. For the lower sign, the saddle point is

$$e^{(k'/2M'-t/4M')} \approx -1/2,$$

which, to zeroth order in ω_k, gives

$$t = 4i\pi M + \text{real}$$

and to first order in ω_k, gives

$$t = 4i\pi(M - \omega_k) + \text{real}.$$

Inserting these values of the saddle point into the integrands gives for the Bogoliubov coefficients,

$$\left|\frac{\alpha_{kk'}}{\beta_{kk'}}\right| = e^{4\pi(M-\omega_k)\omega_k}. \tag{4.50}$$

The flux of radiation from the black hole is given by (4.46),

$$F_\infty(\omega_k) = \frac{d\omega_k}{2\pi}\frac{\Gamma(\omega_k)}{e^{8\pi(M-\omega_k)\omega_k}-1}. \tag{4.51}$$

There is an alternative way of viewing the saddle point calculation, which provides additional insight into the physical origin of the radiation. Let us rewrite the integral (4.49) as

$$\int_{-\infty}^{\infty} dt e^{i\omega_k t \pm iS(t,2(M+\omega_{k'}))}. \tag{4.52}$$

The saddle point is given by that value of t for which the derivative of the expression in the exponent vanishes:

$$\omega_k \pm \frac{\partial S}{\partial t}(t, 2(M+\omega_k)) = 0.$$

But $\partial S/\partial t$ is just the negative of the Hamiltonian,

$$\frac{\partial S}{\partial t} = p_t = M - M_+$$

so the saddle point equation becomes

$$M_+ = M \pm \omega_k.$$

To find the corresponding values of t, we insert this relation into (4.37) and (4.38):

$$t = 4(M \pm \omega_k) \log \left[\frac{\sqrt{2(M+\omega_k)+\epsilon} - \sqrt{2(M+\omega_k)}}{\sqrt{\hat{r}(0)} - \sqrt{2(M \pm \omega_k)}} \right] \tag{4.53}$$

$$k = -\hat{r}(0) \log \left[\frac{\sqrt{\hat{r}(0)} - \sqrt{2(M \pm \omega_k)}}{\sqrt{\hat{r}(0)} - \sqrt{2M}} \right], \tag{4.54}$$

where we have written $\hat{r} = 2(M+\omega_k) + \epsilon$ to make explicit that $\hat{r}$ must lie outside the point where the solutions $u_k(r)$ break down. We desire to solve for t as $k \to \infty$. For the upper choice of sign, we find from (4.54) that

$$\sqrt{\hat{r}(0)} = \sqrt{2(M+\omega_k)} + \mathrm{O}\left(e^{-k/2M}\right),$$

which, from (4.53), then shows that the corresponding value of t is purely real.

For the lower choice of sign we have,

$$\sqrt{\hat{r}(0)} = \sqrt{2(M-\omega_k)} - \mathrm{O}\left(e^{-k/2M}\right).$$

Continuing t into the upper half plane, we find from (4.53) that

$$t = 4i\pi(M - \omega_k) + \text{ real}.$$

These results of course agree with our previous findings.

The preceding derivation invites us to interpret the radiation as being due to negative energy particles propagating in imaginary time. The particles originate from just inside the horizon, and cross to the outside in an imaginary time interval $4\pi(M - \omega_k)$. This, perhaps, helps clarify the analogy between black hole radiance and pair production in an electric field, which, in an instanton approach [32], is also calculated by considering particle trajectories in imaginary time.

Finally, let us return to the question of thermality. One might have guessed that the correct exponential suppression factor could be the Boltzmann factor for nominal temperature corresponding to the mass of the hole before the radiation, after the radiation, or somewhere in between. Thus one might have guessed that the exponential suppression of the radiance could take the form $e^{-\omega/T_{\text{before}}}$, $e^{-\omega/T_{\text{after}}}$, or something in between. Our result, to lowest order, corresponds to the nominal temperature for emission being equal to T_{after}.

4.2 Corrections to Charged Black Hole Radiance

In this section, two additional things are done. First, we extend the calculations to include a charged black hole, and charged matter. Although this step does not present any significant formal difficulties, the physical results we obtain are considerably richer than what we found in our previous calculations involving neutral holes and shells. In the neutral case the final result could be summarized as a simple replacement of the nominal temperature governing the radiation by the Hawking temperature for the mass after radiation, so that the "Boltzmann factor" governing emission of energy ω from a hole of mass M became

$$e^{-\omega/T_{\text{eff.}}} = e^{-\omega 8\pi(M-\omega)} . \tag{4.55}$$

Note that the argument of the exponential is *not* simply proportional to the energy ω, so that the spectrum is not, strictly speaking, thermal. While the deviation from thermality is important in principle its structure, in this case, is rather trivial, and one is left wondering whether that is a general result. Fortunately we find that for charged holes the final results are much more complex. We say "fortunately", not only because this relieves us of the nagging fear that we have done a simple calculation in a complicated way, but also for more physical reasons. For one knows on general grounds that the thermal description of black hole radiance breaks down completely for near-extremal holes. One might anticipate, therefore, that something more drastic than a simple modification of the nominal temperature will occur – as indeed we find. A particularly gratifying consequence of the accurate formula is a form of "quantum cosmic censorship". Whereas a literal application of the conventional thermal formulas for radiation yields a non-zero amplitude for radiation past extremality – that is, radiation leaving behind a hole with larger charge than mass – we find (within our approximations) *vanishing* amplitude for such processes.

Second, we discuss in a more detailed fashion the relationship between our method of calculation, which proceeds by reduction to an effective particle theory, and more familiar approximations. We show that it amounts to saturation of the functional integral of the underlying s-wave field theory with one-particle intermediate states, or alternatively to neglect of vacuum polarization. It is therefore closely related to conventional eikonal approximations. We demonstrate the reduction of the field theory to a particle theory explicitly in the related problem of particle creation by a strong spherically symmetric charge source, which is a problem of independent interest.

4.2.1 Self-Interaction Correction

Our system consists of a matter shell of rest mass m and charge q interacting with the electromagnetic and gravitational fields. The corresponding action is

$$S = \int [-m\sqrt{-\hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu} + q\hat{A}_\mu d\hat{x}^\mu] + \frac{1}{16\pi}\int d^4x\sqrt{-g}\,[\mathcal{R} - F_{\mu\nu}F^{\mu\nu}] \quad (4.56)$$

The gravitational contribution to the Hamiltonian is the same as in (4.6), and the shell and electromagnetic contributions are

$$\mathcal{H}_t^s = \left(\sqrt{(p/\hat{L})^2 + m^2} - q\hat{A}_t\right)\delta(r - \hat{r}) \quad ; \quad \mathcal{H}_r^s = -p\,\delta(r - \hat{r}) \quad (4.57)$$

$$\mathcal{H}_t^{EM} = \frac{N^t L {\pi_{A_r}}^2}{2R^2} - A_t\,\pi'_{A_r} \quad (4.58)$$

To arrive at this form we have chosen a gauge such that A_t is the only non-vanishing component of A_μ. Of course, we set $A_r = 0$ only *after* computing the canonical momentum π_{A_r}.

Constraints are found by varying the action with respect to N^t, N^r, and A_t,

$$\mathcal{H}_t \equiv \mathcal{H}_t^s + \mathcal{H}_t^G + \mathcal{H}_t^{EM} = 0 \quad ; \quad \mathcal{H}_r \equiv \mathcal{H}_r^s + \mathcal{H}_r^G = 0$$
$$\pi'_{A_r} + q\,\delta(r - \hat{r}) = 0. \tag{4.59}$$

π_R can be eliminated by forming the linear combination of constraints

$$0 = \frac{R'}{L}\mathcal{H}_t + \frac{\pi_L}{RL}\mathcal{H}_r = -\mathcal{M}' + \frac{R'}{L}(\mathcal{H}_t^s + \mathcal{H}_t^{EM}) + \frac{\pi_L}{RL}\mathcal{H}_r^s \tag{4.60}$$

where

$$\mathcal{M} = \frac{{\pi_L}^2}{2R^2} + \frac{R}{2} - \frac{RR'^2}{2L^2}. \tag{4.61}$$

We see from the Gauss' law constraint that $-\pi_{A_r}(r)$ is the charge contained within a sphere of size r, so we define: $Q(r) \equiv -\pi_{A_r}(r)$

Now, if the shell was absent $\mathcal{M}(r)$ and $Q(r)$ would be given by

$$\mathcal{M}(r) = M - \int_r^\infty dr\,\frac{R'(r)\mathcal{H}_t^{EM}(r)}{L(r)} \quad ; \quad Q(r) = Q \tag{4.62}$$

with M and Q being the mass and charge of the black hole as seen from infinity. In the gauge $L = 1, R = r$ these become

$$\mathcal{M}(r) = M - Q^2/2r \quad ; \quad Q(r) = Q. \tag{4.63}$$

With the shell present we retain the expression (4.63) for the region inside the shell, $r < \hat{r}$, whereas outside the shell we write (with $L = 1,\ R = r$),

$$\mathcal{M}(r) = M_+ - (Q + q)^2/2r \quad ; \quad Q(r) = Q + q \tag{4.64}$$

where M_+ and $Q + q$ are the mass and charge of the hole-shell system as measured at infinity.

By using the constraints we can determine π_R, π_L, and an expression for M_+, in terms of the shell variables. These relations can then be inserted in the action to give an effective action depending only on the shell variables.

The calculation for the present case runs precisely parallel to the uncharged case, resulting in

$$S = \int dt \left[\dot{\hat{r}} \left(\sqrt{2M\hat{r} - Q^2} - \sqrt{2M_+\hat{r} - (Q+q)^2} \right) \right.$$
$$\left. -\eta\dot{\hat{r}}\hat{r} \log \left| \frac{\sqrt{\hat{r}} - \eta\sqrt{M_+ - (Q+q)^2/2\hat{r}}}{\sqrt{\hat{r}} - \eta\sqrt{M - Q^2/2\hat{r}}} \right| - M_+ \right] \tag{4.65}$$

where $\eta \equiv \operatorname{sgn}(p)$, and we have now specialized to a massless shell ($m = 0$). The canonical momentum is then

$$p_c = \sqrt{2M\hat{r} - Q^2} - \sqrt{2M_+\hat{r} - (Q+q)^2} - \eta\hat{r} \log \left| \frac{\sqrt{\hat{r}} - \eta\sqrt{M_+ - (Q+q)^2/2\hat{r}}}{\sqrt{\hat{r}} - \eta\sqrt{M - Q^2/2\hat{r}}} \right|. \tag{4.66}$$

We need wish to find the short wavelength solutions which are accurately described by the WKB approximation. Writing these solutions as $v(t,r) = e^{iS(t,r)}$ with S rapidly varying, we can make the replacements

$$p_c \to \frac{\partial S}{\partial r} \quad ; \quad M_+ - M \to \frac{\partial S}{\partial t}.$$

$S(t,r)$ satisfies the Hamilton-Jacobi equation, and so is found by computing classical action along classical trajectories. We first choose the initial conditions for $S(t,r)$ at $t = 0$:

$$S_k^q(0,r) = kr. \tag{4.67}$$

We have a appended a subscript and a superscript to denote the initial condition and charge of the solution. The corresponding classical trajectory has the initial condition $p_c = k$ at $t = 0$. $S_k^q(t,r)$ is then given by

$$S_k^q(t,r) = k\hat{r}(0) + \int_{\hat{r}(0)}^{r} d\hat{r}\, p_c(\hat{r}) - (M_+ - M)t. \tag{4.68}$$

To determine the radiance from the hole we will will only need to consider the behaviour of the solutions near the horizon. Furthermore, only the most rapidly varying part of the solutions will contribute to the late-time radiation.

With this in mind, we can write the momentum as (choosing $\eta = 1$ for an outgoing solution)

$$p_c(\hat{r}) \approx -\hat{r} \log \left| \frac{\hat{r} - R_+(M_+, Q+q)}{(\hat{r} - R_+(M,Q))(\hat{r} - R_-(M,Q))} \right| \tag{4.69}$$

so that the initial condition becomes

$$k = -\hat{r}(0) \log \left| \frac{(\hat{r}(0) - R_+(M_+, Q+q))(\hat{r}(0) - R_-(M_+, Q+q))}{(\hat{r}(0) - R_+(M,Q))(\hat{r}(0) - R_-(M,Q))} \right|. \tag{4.70}$$

Similarly, the classical trajectory emanating from $\hat{r}(0)$ is given by approximately,

$$t \approx \frac{2}{R_+(M_+, Q+q) - R_-(M_+, Q+q)} \left[R_+(M_+, Q+q)^2 \log \left| \frac{\hat{r} - R_+(M_+, Q+q)}{\hat{r}(0) - R_+(M_+, Q+q)} \right| \right.$$
$$\left. -R_-(M_+, Q+q)^2 \log \left| \frac{\hat{r} - R_-(M_+, Q+q)}{\hat{r}(0) - R_-(M_+, Q+q)} \right| \right]. \tag{4.71}$$

These trajectories are in fact null geodesics of the metric

$$ds^2 = -dt^2 + (dr + \sqrt{2M_+/r - Q^2}\, dt)^2. \tag{4.72}$$

The relations (4.70) and (4.71) allow us to determine M_+ and $\hat{r}(0)$ in terms of the other variables, so that after integrating (4.68) we can obtain an expression for $S_k^q(t,r)$ as a function of k, t, and r.

We can now write down an expression for the field operator:

$$\hat{\phi}(t,r) = \int dk \, [\hat{a}_k v_k^q(t,r) + \hat{b}_k^\dagger v_k^{-q}(t,r)^*]. \tag{4.73}$$

The modes $v_k^q(t,r)$ are nonsingular at the horizon, and so the state of the field is taken to be the vacuum with respect to these modes:

$$\hat{a}_k \left|0_v\right\rangle = \hat{b}_k \left|0_v\right\rangle = 0.$$

Alternatively, we can consider modes which are positive frequency with respect to the Killing time t. We write these modes as $u_k^q(r)e^{-i\omega_k t}$ where the $u_k^q(r)$ are singular at the horizon, $r = R_+(M + \omega_k, Q+q)$. Then

$$\hat{\phi}(t,r) = \int dk \, [\hat{c}_k u_k^q(r) e^{-i\omega_k t} + \hat{d}_k^\dagger u_k^{-q}(r)^* e^{i\omega_k t}]. \tag{4.74}$$

The two sets of operators are related by Bogoliubov coefficients,

$$\hat{c}_k = \int dk \, [\alpha_{kk'} \hat{a}_{k'} + \beta_{kk'} \hat{b}^\dagger_{k'}]. \tag{4.75}$$

From (4.73, 4.74) $\alpha_{kk'}$ and $\beta_{kk'}$ are found to be

$$\alpha_{kk'} = \frac{1}{2\pi u^q_k(r)} \int_{-\infty}^{\infty} dt \, e^{i\omega_k t} v^q_{k'}(t, r)$$

$$\beta_{kk'} = \frac{1}{2\pi u^q_k(r)} \int_{-\infty}^{\infty} dt \, e^{i\omega_k t} v^{-q}_{k'}(t, r)^*. \tag{4.76}$$

Here, r is taken to be slightly outside the horizon, $r = R_+(M + \omega_k, Q + q) + \epsilon$. These coefficients can be evaluated in the saddle point approximation. Recalling that $v^q_k(t, r) = e^{iS^q_k(t,r)}$, the saddle point equation for $\alpha_{kk'}$ becomes

$$\omega_k = -\frac{\partial S^q_{k'}}{\partial t} = M^q_+ - M. \tag{4.77}$$

This leads to a purely real value of t for the saddle point. For $\beta_{kk'}$ we have

$$\omega_k = \frac{\partial S^{-q}_{k'}}{\partial t} = M - M^{-q}_+. \tag{4.78}$$

From (4.70, 4.71) we find that the saddle point value for t has an imaginary part given by

$$\mathrm{Im}(t_s) = \frac{2\, R_+(M - \omega_k, Q - q)^2}{R_+(M - \omega_k, Q - q) - R_-(M - \omega_k, Q - q)} \, \pi = \frac{1}{2\, T(M - \omega_k, Q - q)}. \tag{4.79}$$

Therefore,

$$|\beta_{kk'}/\alpha_{kk'}| = \frac{1}{|2\pi u_k(r)|} \exp\Big(\omega_k / T(M - \omega_k, Q - q) + \mathrm{Im}[S^{-q}_{k'}(t_s)^*]\Big). \tag{4.80}$$

The terms in $S^{-q}_{k'}$ which contribute to the second term in the exponent are

$$\int_{\hat{r}(0)}^{r} d\hat{r} \, p_c(\hat{r}) + \omega_k \, \mathrm{Im}(t_s).$$

Using (4.69-4.71) this can be evaluated to give

$$\mathrm{Im}[S_{k'}^{-q}(t_s)^*] = \frac{M\omega + \sqrt{M^2-Q^2}\left(\sqrt{(M-\omega)^2-(Q-q)^2} - \sqrt{M^2-Q^2}\right)}{2\,T(M-\omega,Q-q)\,R_+(M,Q)} \tag{4.81}$$

resulting in

$$\left|\frac{\beta_{kk'}}{\alpha_{kk'}}\right|^2 = \exp\left(-\frac{\sqrt{M^2-Q^2}\,[\omega - \sqrt{(M-\omega)^2-(Q-q)^2} + \sqrt{M^2-Q^2}\,]}{T(M-\omega,Q-q)\,R_+(M,Q)}\right). \tag{4.82}$$

This is the effective Boltzmann factor governing emission. Sufficiently far from extremality, when ω, $q \ll \sqrt{M^2-Q^2}$, we can expand (4.82) to give

$$\left|\frac{\beta_{kk'}}{\alpha_{kk'}}\right|^2 \approx \exp\left(-\frac{\omega - \frac{Qq}{R_+(M,Q)} + \frac{M^2q^2+Q^2\omega^2-2MQ\omega q}{2(M^2-Q^2)R_+(M,Q)}}{T(M-\omega,Q-q)}\right) \tag{4.83}$$

as compared to the free field theory result [3],

$$\left|\frac{\beta_{kk'}}{\alpha_{kk'}}\right|^2 = \exp\left(-\frac{\omega - \frac{Qq}{R_+(M,Q)}}{T(M,Q)}\right). \tag{4.84}$$

Near extremality, the self-interaction corrections cause the emission to differ substantially from (4.84).

We might ask whether it is possible to reach extremality after a finite number of emissions. Since $T(M-\omega,Q-q)$ appears in the denominator of the exponent of (4.82), the transition probability to the extremal state is in fact zero. We can also ask whether there are transitions to a meta-extremal ($Q > M$) hole. This would have rather dramatic implications as the meta-extremal hole is a naked singularity. To address this question we return to the saddle point equation (4.78). When $Q > M$, R_+ and R_- become complex. From (4.70) we see that a saddle point solution would require that k be complex, but we do not allow this since a complete family of initial conditions $S_k(0,r) = kr$ was defined with k real. Therefore, in the saddle point approximation the extremal hole is stable.

Modes with $|\beta/\alpha| > 1$ formally require larger amplitudes for higher occupation numbers, and thus require special interpretation. Considering for simplicity the free field form of these coefficients, (4.84), we see that such modes occur when $\omega < qQ/R_+$, that is when the incremental energy gain from discharging the Coulomb field overbalances the cost of creating the charged particle. Under these conditions one has dielectric breakdown of the vacuum, just as for a uniform electric field in empty space. Since this physics is not our primary concern in the present work, we shall restrict ourselves to a few remarks. The occupation factor appearing in the formula for radiation in these "superradiant" modes is negative, but the reflection probability exceeds unity, so the radiation flux is positive as it should be. And in general the formulas for physical quantities will appear sensible, although Fock space occupation numbers are not. We can avoid superradiance altogether by considering a model with only massive charged fundamental particles, and holes with a charge/mass ratio small compared to the minimal value for fundamental quanta.

Another interesting variant is to consider a *magnetically* charged hole interacting with neutral matter. In that case, one simply puts $q = 0$ in the fomulae above (but $Q \neq 0$). One could also consider the interaction of dyonic holes with charged matter, and other variants (*e.g.* dilaton black holes) but we shall not do that here.

4.3 Discussion

We have arrived at our results by what may have appeared to be a somewhat circuitous route. Inspired by a field theory question, we calculated the solutions of a single self-gravitating particle at the horizon, and then passed back to field theory by interpreting the solutions as the modes of a second quantized field operator. In this section we hope to clarify the logic of this procedure, and show that it is both correct and efficient, by demonstrating how a single particle action emerges from the truncation of a complete field theory.

We can illustrate this explicitly if we consider the simpler model of spherically symmetric electromagnetic and charged scalar fields interacting in flat space. Our goal is to show that the propagator for the scalar field can be expressed as a Hamiltonian path integral for a single charged shell. To achieve this, two important approximations will be made. The first is that the effects of vacuum polarization will be assumed to be small, so we can ignore scalar loop diagrams. The second is to assume that the dominant interactions involve soft photons, so that the difference in the scalar particle's energy before and after emission or absorption of a photon is small compared to the energy itself. Thus we expect that our expression will be valid for cases where the scalar particle has a large energy, so that the energy transfer per photon is relatively small, and is far from the origin, so that the classical electromagnetic self energy of the particle is a slowly varying function of the radial coordinate. Field theory in this domain is in fact well described by the eikonal approximation, which implements the same approximations we have just outlined. What follows is then essentially a Hamiltonian version of the eikonal method.

We start from the action

$$\begin{aligned} S &= -\frac{1}{4\pi}\int d^4x \left[(\partial_\mu - iqA_\mu)\phi^*\,(\partial^\mu + iqA^\mu)\phi + m^2\phi^*\phi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \\ &= \int dt\,dr\left[\pi_{\phi^*}\dot{\phi}^* + \pi_\phi\dot{\phi} - \left(\frac{\pi_{\phi^*}\pi_\phi}{r^2} + r^2\phi^{*\prime}\phi' + m^2r^2\phi^*\phi + \frac{{\pi_{A_r}}^2}{2r^2}\right)\right. \\ &\qquad\left. -A_t\left(iq[\pi_{\phi^*}\phi^* - \pi_\phi\phi] - \pi'_{A_r}\right)\right]. \end{aligned} \tag{4.85}$$

Defining the charge density

$$\rho(r) \equiv iq[\pi_{\phi^*}(r)\phi^*(r) - \pi_\phi(r)\phi(r)] \tag{4.86}$$

the solution of the Gauss' law constraint is

$$Q(r) \equiv -\pi_{A_r} = \int_0^r dr\,\rho(r) \tag{4.87}$$

and so the scalar field Hamiltonian becomes

$$H = \int_0^\infty dr\left[\frac{\pi_{\phi^*}\pi_\phi}{r^2} + r^2\phi^{*\prime}\phi' + m^2\phi^*\phi + \frac{Q(r)^2}{2r^2}\right]. \tag{4.88}$$

The fields are now written as second quantized operators:

$$\begin{aligned} \hat{\phi} &= \int \frac{dk}{\sqrt{2\pi\,2\omega_k}}\,\frac{[\hat{a}_k e^{ikr} + \hat{b}_k^\dagger e^{-ikr}]}{r} \\ \hat{\pi}_\phi &= i\int \frac{dk}{\sqrt{2\pi}}\sqrt{\frac{\omega_k}{2}}\,r\,[\hat{a}_k^\dagger e^{-ikr} - \hat{b}_k e^{ikr}] \end{aligned} \tag{4.89}$$

where $\omega_k = \sqrt{k^2 + m^2}$, and we also have $\hat{\phi}^* = \hat{\phi}^\dagger$, $\hat{\pi}_{\phi^*} = \hat{\pi_\phi}^\dagger$. To ensure that the field is nonsingular at the origin we impose the conditions $\hat{a}_{-k} = -\hat{a}_k$, $\hat{b}_{-k} = -\hat{b}_k$, and take the limits of all k integrals to be from $-\infty$ to ∞.

We now write the Hamiltonian in terms of the creation and annihilation operators. In doing so we shall normal order the operators, which corresponds to omitting vacuum polarization since we do not allow particle-antiparticle pairs to be created out of the vacuum. Also when evaluating ϕ' we shall use the geometrical optics approximation, $(e^{ikr}/r)' \approx ike^{ikr}/r$, valid for $k \gg 1/r$. Then the quadratic part of the Hamiltonian becomes,

$$\int_0^\infty dr \left[\frac{\hat{\pi}_{\phi^*}\hat{\pi}_\phi}{r^2} + r^2 \hat{\phi}^{*\prime}\hat{\phi}' + m^2 \hat{\phi}^* \hat{\phi} \right] = \frac{1}{2} \int dk\, \omega_k [\hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k]. \tag{4.90}$$

Next we consider the interaction term. When evaluating this there will arise factors of $\sqrt{\omega_{k'}/\omega_k}$. The essence of the soft photon approximation is that we replace these factors by 1, since we are assuming that $\Delta\omega/\omega \ll 1$ for the emission or absorption of a single photon. Then, after normal ordering, we can evaluate the charge density to be:

$$\hat{\rho}(r) = q \int \frac{dk\, dk'}{2\pi} [\hat{a}_k^\dagger \hat{a}_{k'} - \hat{b}_k^\dagger \hat{b}_{k'}] e^{i(k-k')r}. \tag{4.91}$$

We now wish to calculate matrix elements of the Hamiltonian between one particle states. A basis of one particle states labelled by position is given by

$$|r\rangle = \int \frac{dk}{\sqrt{2\pi}}\, e^{-ikr}\, \hat{a}_k^\dagger |0\rangle. \tag{4.92}$$

The free part of the Hamiltonian then has matrix elements

$$\langle r_2 | \frac{1}{2} \int dk\, \omega_k [\hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k] | r_1 \rangle = \int \frac{dk}{2\pi}\, \omega_k [e^{ik(r_2 - r_1)} - e^{ik(r_2 + r_1)}]. \tag{4.93}$$

The second term in the brackets corresponds to the path from r_1 to r_2 which passes through the origin. These paths will not contribute to local processes far from the origin, so we drop this term. The matrix elements of the interaction term for closely spaced points r_1 and r_2 are:

$$\langle r_2 | \int_0^\infty dr\, \frac{\hat{Q}(r)^2}{2r^2} | r_1 \rangle = \frac{q^2}{2r_1} \int \frac{dk}{2\pi} e^{ik(r_2 - r_1)}. \tag{4.94}$$

Putting these together, we find the matrix elements of the Hamiltonian,

$$\langle r_2|\hat{H}|r_1\rangle = \int \frac{dk}{2\pi}\, e^{ik(r_2-r_1)}(\sqrt{k^2+m^2} + q^2/2r_1). \tag{4.95}$$

Now we can follow the standard route which leads from matrix elements of the Hamiltonian to a path integral expression for the time evolution operator, with the result

$$\langle r_f|e^{-i\hat{H}t}|r_i\rangle = \int_{r(0)=r_i}^{r(t)=r_f} \mathcal{D}p\,\mathcal{D}r\; e^{i\int_0^t dt'\,(p\dot{r}-\sqrt{p^2+m^2}-q^2/2r)}. \tag{4.96}$$

The action in the exponent is precisely that of a charged shell, with $q^2/2r$ being the electromagnetic self energy.

We now discuss how this analysis can be applied to the case where we include gravitational interactions. The resulting field Hamiltonian is much more complex, and so we will not be able to explicitly calculate the effective shell action. However, the preceding derivation allows us to argue that were we to do so, we would simply derive the effective action obtained in section 2. The nature of the black hole radiance calculation makes us believe that the approximations used to arrive at a shell action are justified. This is so because for a large ($M \gg m_p$) hole the relevant field configurations are short wavelength solutions moving in a region of relatively low curvature, and these are the conditions which we argued make the eikonal approximation valid.

For simplicity, we will consider an uncharged self-gravitating scalar field. If we truncate to the s-wave we arrive at what is known as the BCMN model, originally considered in [17] and corrected in [26]. The action is

$$\begin{aligned} S &= \frac{1}{4\pi}\int d^4x\,\sqrt{-g}\left[\frac{1}{4}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \\ &= \int dt\,dr\,\left[\pi_\phi\dot{\phi} + \pi_R\dot{R} + \pi_L\dot{L} - N^t(\mathcal{H}_t^\phi + \mathcal{H}_t^G) - N^r(\mathcal{H}_r^\phi + \mathcal{H}_r^G)\right] - \int dt\, M_{ADM} \end{aligned} \tag{4.97}$$

with

$$\mathcal{H}_t^\phi = \frac{1}{2}\left(\frac{{\pi_\phi}^2}{LR^2} + \frac{R^2}{L}\phi'^2\right) \quad ; \quad \mathcal{H}_r^\phi = \pi_\phi\phi'. \tag{4.98}$$

The analog of (4.60) is now

$$\mathcal{M}' = \frac{R'}{L}\mathcal{H}_t^\phi + \frac{\pi_L}{RL}\mathcal{H}_r^s = \frac{R'}{2L^2}\left(\frac{{\pi_\phi}^2}{R^2} + R^2\phi'^2\right) + \frac{\pi_L\pi_\phi\phi'}{RL} \tag{4.99}$$

The Hamiltonian is

$$H = M_{ADM} = \mathcal{M}(\infty) = M + \int_0^\infty dr \left[\frac{R'}{L}\mathcal{H}_t^\phi + \frac{\pi_L}{RL}\mathcal{H}_r^s \right]. \qquad (4.100)$$

To obtain an expression for H which depends only on ϕ and π_ϕ we must choose a gauge and solve the constraints. We can obtain an explicit result if we choose the gauge $R = r$, $\pi_L = 0$. Then, defining

$$h(r) \equiv \frac{1}{2}\left(\frac{{\pi_\phi}^2}{r^2} + r^2 \phi'^2 \right), \qquad (4.101)$$

L is determined from (4.99),

$$\mathcal{M}'(r) = \left(\frac{r}{2} - \frac{r}{2L^2} \right)' = \frac{h(r)}{L^2} \qquad (4.102)$$

so

$$\frac{1}{L^2} = -\frac{2M}{r} e^{-2\int_0^r dr'\, h(r')/r'} + \frac{1}{r} e^{-2\int_0^r dr'\, h(r')/r'} \int_0^r dr'\, e^{2\int_0^{r'} dr''\, h(r'')/r''} \qquad (4.103)$$

which then leads to

$$H = M e^{-2\int_0^\infty dr\, h(r)/r} + \int_0^\infty dr\, h(r) e^{-2\int_r^\infty dr'\, h(r')/r'}. \qquad (4.104)$$

This generalizes the result of [26] to include a nonzero mass M for the pure gravity solution. To make a direct comparison with our work in the previous section, it would be preferable to obtain the Hamiltonian in $L = 1, R = r$ gauge. This is more difficult and we do not know the explicit expression. For the moment, though, we are mainly interested in the qualitative structure of the Hamiltonian, and (4.104) will be sufficient for our purposes. The various nonlocal terms contained in the Hamiltonian (4.104) correspond to gravitons attaching onto the particle's worldline. If we expand the exponentials in (4.104), we see that there arise an infinite series of bi-local, tri-local, ..., terms resulting from the non-linearity of gravity. Now we could, in principle, repeat the analysis which led to an effective shell action for the charged field in flat space. In that case the calculation could be done with only modest effort because there was only a single quartic interaction term. In the present case we would have to sum the infinite series of terms that arise; our point is that handling all of these terms is cumbersome, to say the least, and that it is much simpler to proceed as in section 2.

4.4 Multi-particle Correlations

We have seen how to obtain the self-interaction correction to the probability of single particle emission. The natural next step would be to obtain similar results for multi-particle processes, involving some combination of ingoing and outgoing particles. The ultimate goal, of course, it to construct a complete S-matrix relating arbitrary in and out states. This brings to the fore what is usually regarded as the central conceptual puzzle of black hole physics: does such an S-matrix exist which unitarily relates states described on $\mathcal{J}^+$ and $\mathcal{J}^-$, or is there "information loss" is the sense that pure states on $\mathcal{J}^-$ can evolve into mixed states on $\mathcal{J}^+$? While the latter possibility clearly violates a tenet of quantum physicists, there is at present no satisfactory proposal as to how the former possibility can be realized.

If information is preserved in gravitational collapse, and an S-matrix exists, it will require the existence of intricate correlations between particles on $\mathcal{J}^+$ which encode the details of the state on $\mathcal{J}^-$. An understanding of how this situation might arise is currently precluded by the fact that essentially nothing is known about how to compute *any* correlations on $\mathcal{J}^+$, much less those which would preserve all information.

In free field theory, the state on $\mathcal{J}^+$ is described by an exactly thermal density matrix, and no one knows how to go beyond the free field approximation, except in the case of the single particle processes we have been discussing. However, we can envision calculating the correlations between two emitted particles by an extension of our previous methods, simply including two shells instead of one. Presumably, the correlations would be quite complicated in the case of short time separation between the particles, but upon going to to large time separations one would see the later particle being emitted with a probability corresponding to a hole of mass $M-\omega$, where ω is the energy of the earlier particle. In addition, there is the possibility for the phases of the particles to be correlated even for large time separation, owing to the fact that the two shells were closely spaced near the horizon at early times.

There is no obstacle to writing fown the action for two shells,

$$S = \int dt\, [p_1 \dot{r_1} + p_2 \dot{r_2} - H(r_1, r_2, p_1, p_2)]. \tag{4.105}$$

H is again given by the ADM mass, which can be expressed in terms of the shell variables by solving the constraints. The action $S_{k_1 k_2}(t, r_1, r_2)$ can then be computed for the initial condition

$$S_{k_1 k_2}(0, r_1, r_2) = k_1 r_1 + k_2 r_2 \tag{4.106}$$

by integrating the Hamilton-Jacobi equation in precisely the same manner as for the one particle case. Presumably, all correlations between the particles are encoded in this action; unfortunately, we do not know the code, for reasons that we now discuss.

In the single particle case it was possible to make progress because we know how to pass from the first quantized description in terms of $S_k(t,r)$, to a field description; namely by writing

$$\phi(t,r) = \sum_k [a_k e^{iS_k(t,r)} + a_k^\dagger e^{-iS_k(t,r)}]. \tag{4.107}$$

This then put the full power of Bogoliubov transformations at our disposal, which enabled us to determine the emission probabilities. By constrast, in the multi-particle case the relation between the first quantized description and the field description is unclear — we are not aware of any extension of (4.107) which takes into account the two particle solutions. This represents a major obstacle, since a proper interpretation of the theory requires that we have a field description. On the other hand, it seems most likely that this problem is merely a technical one, as there is no reason to believe that the theory is ill defined or inconsistent.

Chapter 5

Black Hole Entropy

Almost all researchers agree that a black hole has an entropy equal to one quarter the area of its event horizon, even though there is no consensus as to what the entropy represents physically. The most appealing possibility is that the entropy counts the number of black hole microstates, *i.e.* the black hole Hamiltonian has $e^{S(M)}$ eigenvalues between 0 and M. Since it may seem rather remarkable that this could be deduced from the study of free field theory on a classical background geometry, we will attempt put the argument in perspective by recalling the corresponding derivation in ordinary thermodynamics. Suppose we wish to determine the entropy of some body of matter. To proceed, we would first perform the experiment of placing the body in contact with a heat bath of temperature T, waiting for equilibrium to occur, and then measuring $E(T)$, the energy of the body. Then, assuming that equilibrium occurs when the total number of states of the system is maximized, we have

$$\frac{\partial S_{\text{body}}(E)}{\partial E} = \frac{\partial S_{\text{bath}}}{\partial E}. \tag{5.1}$$

Finally, since $\partial S_{\text{bath}}/\partial E = 1/T$, we obtain

$$S_{\text{body}}(E) = \int_0^E \frac{dE}{T(E)}. \tag{5.2}$$

Clearly, the derivation relies on two features: a) our ability to measure $E(T)$, or $T(E)$, b) the assumption that in equilibrium the number of states is maximized. Now let is return to the black hole case. Obviously, no one has performed an experiment to see whether a black hole of mass M is in equilibrium with a heat bath of temperature $1/8\pi M$. The point is that it appears that any nonsingular state of the black hole satisfies this property. We saw that by requiring regularity of the stress energy tensor at the horizon we were inevitably led to consider states which radiate at the Hawking temperature. The status of point (b) is much less clear. Just because the hole is in equilibrium with the heat bath we cannot conclude that the black hole is making frequent transitions among its microstates, and that the probability distribution of these states is Boltzmann. A rather trivial example of this behaviour is provided by a perfectly reflecting mirror; it can certainly be in equilibrium with thermal radiation without satisfying these other conditions. In fact, we shall se shortly that it does not seem to be possible to assign a Boltzmann probability distribution to the microstates of the hole. Nevertheless, let us assume for now that assumption (b) is justified. Then,

$$S_{\text{hole}}(M) = \int_0^M \frac{dM}{1/8\pi M} = 4\pi M^2 = \frac{A}{4}. \tag{5.3}$$

The belief that the entropy does count microstates is bolstered by the existence of a completely unrelated derivation due to Gibbons and Hawking [33]. We will not go through the details but simply sketch the idea. Their approach relies on the fact that the partition function for a system can be expressed as a path integral over configurations periodic in imaginary time:

$$Z = \operatorname{Tr} e^{-\beta H} = \int_{\text{periodic}} \mathcal{D}g_{\mu\nu} e^{-S}. \tag{5.4}$$

Here, the path integral is over Euclidean metrics with periodicity $\Delta\tau_E = i\beta$. One then calculates the path integral in saddle point approximation and notes that only the analytic continuation of the black hole geometry with $\beta = 8\pi M$ contributes, because other geometries have a conical singularity at the horizon. After calculating the action and subtracting the flat space contribution, they obtain:

$$Z(\beta) = e^{A/4} e^{-\beta M} \tag{5.5}$$

leading to the identification $S = A/4$.

This derivation also has serious problems, not least of which is the fact that the Euclidean path integral for gravity may not even exist due to the well known conformal instability. Nevertheless, it has the virtue that the answer agrees with the previous result. In any event, it seems overwhelmingly likely that the entropy has some sort of deep significance, and the most natural interpretation is that it counts microstates.

5.1 State Counting

If we are inclined to believe that a black hole has the enormous number of states e^S, it behooves us to explain how these states can be understood in terms of the underlying Hamiltonian. It would be most satisfying if quantization of this Hamiltonian revealed a discrete set of states whose number could be counted to yield the entropy. In this section we will describe some efforts along these lines, and the divergence problems [34] which ensue. In keeping with the spirit of this thesis, the discussion will be confined to spherically symmetric configurations. While we have no particular reason to believe that all of the states can be accounted for by considering only the s-wave, the sorts of problems that arise seem to afflict the higher partial waves in the same way. Pure gravity in the s-wave has no dynamical degrees of freedom, and so to have a nontrivial theory to quantize we include a massless scalar field. It is a little unsettling that we are forced to include to matter in order to calculate, since the expression for black hole entropy has no explicit dependence on the matter content of the world. However, it is possible that there *is* matter dependence, but it can be absorbed into the value of Newton's constant which appears in the entropy formula [36, 37].

As a first approach, we can try to proceed in the same way as if we were quantizing a soliton in flat space — by quantizing the quadratic fluctuations of the field about the background solution. Considering only the quadratic fluctuations amounts to doing free field theory. For simplicity, we will work in Schwarzschild coordinates and use modes of definite energy, $e^{-i\omega(t-r_*)}$, where $r_* = r + 2m\log{(r/2M-1)}$, and write:

$$\phi = \sum_{\omega} [a_\omega e^{-i\omega(t-r_*)} + a_\omega^\dagger e^{i\omega(t-r_*)}]. \tag{5.6}$$

We consider only the outgoing modes since they alone lead to the divergence problems. In order to count modes we must make the frequencies discrete in some way. The easiest thing to do is to impose periodic boundary conditions at $r = 2M + \epsilon$ and $r = L$. The ϵ is included because the modes become singular at the horizon, and L is some arbitrarily chosen radius outside the horizon. We take $L \gg 2M \gg \epsilon$. This leads to the allowed frequencies:

$$\omega_n \approx \frac{\pi n}{M \log{(L/\epsilon)}}. \tag{5.7}$$

This means that there are

$$n = \frac{M\omega \log{(L/\epsilon)}}{\pi} \tag{5.8}$$

single particle states for the black hole with energies between M and $M + \omega$. The problem is that there is no physical reason to keep ϵ finite, but the number of states diverges as $\epsilon \to 0$. The reason for this behavior is due to the fact that $r_* \to -\infty$ as $r \to 2M$, so the modes oscillate an infinite number of times before they reach the horizon.

It seems quite possible that the infinite number of oscillations is simply a result of using free field theory, and that once the proper self-interaction corrections are included a finite result will be obtained. To illustrate this, let us recall the expression for the canonical momentum of a self-interacting particle,

$$p_c = \sqrt{2Mr} - \sqrt{2M_+ r} - r \log\left|\frac{\sqrt{r} - \sqrt{2M_+}}{\sqrt{r} - \sqrt{2M}}\right|. \tag{5.9}$$

For an energy eigenstate $M_+ = M + \omega$, and the corresponding mode is

$$\psi(r,t) = e^{i\int^r p_c(r')dr' - i\omega t}. \tag{5.10}$$

Thus the number of oscillations is finite or infinite depending on whether $\int^{2(M+\omega)} p_c(r')dr'$ is finite or infinite. The singular part of p_c as $r \to 2M$ is:

$$p_c \approx -2(M+\omega) \log{[r - 2(M+\omega)]} \tag{5.11}$$

which leads to,

$$\int^{2(M+\omega)} p_c(r')dr' = \text{finite}. \tag{5.12}$$

The (incorrect) free field theory result is recovered by expanding in ω. Then p_c has the singular part

$$p_c = \frac{4M\omega}{r - 2M} \tag{5.13}$$

and $\int^{2M} p_c dr$ is again infinite.

Thus by including self-interaction we seem to have solved the divergence problem. However, there is a major caveat. The mode solutions (5.10) were obtained using the WKB approximation. As we discussed earlier, the WKB approximation for a solution of the form e^{iS} is only valid provided

$$\left|\frac{\partial S}{\partial r}\right| \gg \left|\frac{\partial^2 S}{\partial r^2}\right|^{\frac{1}{2}}, \left|\frac{\partial^3 S}{\partial r^3}\right|^{\frac{1}{3}} \ldots \tag{5.14}$$

In other words, S should be rapidly oscillating but the rate of change of oscillation must not be too large. Applying this condition to (5.10) gives

$$|p_c(r)| \gg \left|\frac{dp_c(r)}{dr}\right|^{\frac{1}{2}} \tag{5.15}$$

or:

$$|2(M+\omega)\log\left[r-2(M+\omega)\right]| \gg \left|\frac{2(M+\omega)}{r-2(M+\omega)}\right|^{\frac{1}{2}}. \tag{5.16}$$

This condition is not satisfied as $r \to 2(M+\omega)$. Therefore, we cannot trust the behavior of (5.10) near the horizon, including the conclusion that it oscillates a finite number of times. The most convincing resolution would be to go beyond the WKB approximation and obtain the correct result for ψ. But, as we have discussed previously, this requires the resolution of factor ordering problems which we do not know how to solve at the present time. The most we can say is that the preceding analysis suggests quite strongly that the divergences are connected with an improper treatment of the self-interaction of the modes near the horizon.

The breakdown of the WKB approximation occurs when we insist on using modes of definite energy. Earlier, we saw how to derive a complete set of modes which were nonsingular at the horizon and are accurately described by the WKB approximation. These were:

$$u_k(t,r) = e^{iS_k(t,r)} \tag{5.17}$$

where:

$$S_k(t,r) = -(2M^2 - r^2/2)\log\left[1 + e^{(k/2M' - t/4M')}\right] \tag{5.18}$$

with M' given by (4.40). The coordinates are those of (3.3). Since these modes do not have definite energy, there is no straightforward way to count states using the microcanonical ensemble. However, we can imagine putting the black hole in contact with a distant heat bath at the Hawking temperature. If the black hole behaved as an ordinary thermodynamic body we could proceed by computing the partition function, and from that extract the entropy by standard thermodynamic formulas. Since the partition function is a trace,

$$Z = \mathrm{Tr}\,(e^{-\beta H}), \tag{5.19}$$

it is independent of which basis we choose to describe the states, and so the modes (5.17) are as good as any other. Our working assumption is that we can describe all the states as fluctuations about a fixed black hole of mass M. Since in the s-wave there are no purely gravitational fluctuations, the Fock space built on the modes (5.17) should provide a complete description of these states. Therefore we can write

$$Z =< 0|e^{-\beta H}|0> + \int dk_1 < k_1|e^{-\beta H}|k_1> + \int dk_1\, dk_2 < k_1 k_2|e^{-\beta H}|k_1 k_2> + \ldots. \tag{5.20}$$

H is the total energy as measured at infinity, so $e^{-\beta H}$ generates translations in imaginary time:

$$e^{\beta H}\phi(t)e^{-\beta H} = \phi(t - i\beta). \tag{5.21}$$

This feature makes Z easy to calculate. The first term gives simply

$$< 0|e^{-\beta H}|0 >= 1 \tag{5.22}$$

by definition of the vacuum. The next term is more interesting,

$$< k|e^{-\beta H}|k >= \frac{1}{2\pi}\int_{-\infty}^{\infty} dr\, u_k^*(0,r)u_k(-i\beta,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr\, e^{-iS_k(0,r)}e^{iS_k(-i\beta,r)}. \tag{5.23}$$

Let us concentrate on the behavior for large k. For $k \gg M$ we have

$$S_k(-i\beta, r) \approx 2M(r-2M)(k/2M' + i\beta/4M') \quad ; \quad M' \approx M + \frac{1}{2}(r-2M). \tag{5.24}$$

This is strictly valid only near the horizon, but that is the only region in which we need the solutions since we can always construct wavepackets localized near the horizon. The point we now wish to stress is that $< k|e^{-\beta H}|k >$ goes to a (nonzero) constant independent of k as $k \to \infty$. The precise value of the constant depends on the form of the wavepackets, but at any rate

$$\int dk < k|e^{-\beta H}|k >= \infty \tag{5.25}$$

so that Z cannot be defined.

There is a simple physical explanation for this behavior. A nonsingular mode has positive energy density for points outside the horizon, and negative energy density for points inside. Now imagine being at fixed radius, r, initially outside the horizon, and letting k increase. As k increases, the total mass inside radius r increases. However, the mass can never increase past $M = 2/r$, since if it did one would be inside the horizon, but the modes have negative energy here and so cannot lead to an increase in mass. So as k goes to infinity, the effect on the geometry simply goes to a constant, which explains why the matrix element of $e^{-\beta H}$ also goes to a constant.

The lesson to be learned from all of this is presumably that a black hole cannot be treated as an ordinary thermodynamic body in the sense that its states are distributed according to a Boltzmann distribution when in equilibrium with a heat bath. There are simply too many low energy states localized near the horizon for this distribution to make sense. Let us point out, though, that this conclusion is based on the assumption that the states are correctly described by a local quantum field theory. This assumption may be incorrect, and the divergences may disappear when the correct theory at short distances, such as string theory, is taken into account. That string theory plays a crucial role in black hole physics is argued in, for example, Ref. [36].

Chapter 6

Black Holes and Quantum Tunneling

The radiation of particles from matter evolving along a classical trajectory has been heavily studied in recent years. Less well studied is the radiation accompanying quantum tunneling from one classically allowed trajectory to another. The following question is of interest: if a matter system impinges upon a potential barrier with a radiation field in a certain state, what is the state of the field given that the matter is subsequently observed to be on the other side of the barrier? A method to answer this question in the context of false vacuum decay in flat space was developed by Rubakov [38] and has been generalized to include gravity as well as topology changing processes [39, 40]. The spectrum of radiation is found by solving an imaginary time Schrödinger equation, the occurrence of which leads to novel features. Instead of solving field equations in real time, one is naturally led to consider propagation on the Euclidean solution interpolating between the two classical trajectories. As phase factors in real time are converted into exponential damping factors in imaginary time, the resulting particle creation can be distinctly different and is accompanied by the systematic supression of excitations present before tunneling.

Given this situation, it is natural to ask how the radiation from black holes might be affected by the presence of tunneling. If we consider a distribution of matter, initally outside its Schwarzschild radius, which tunnels through a potential barrier to form a black hole, the conventional calculation [3] of the radiation does not apply. On the other hand, it would be shocking if the same answer was not obtained for the radiation at late times, as this is thought to depend only on the hole's late time geometry and not on its history at early times. Here we compute the radiation for this process and show that while the Euclidean time evolution has an effect at early times, it has none at late times so that the standard result is in fact obtained.

In order illustrate the technique of Ref. [38] in a simpler setting, we first study the effect of tunneling on another well known radiating system — the moving mirror [28]. We show in Sect. (2) how an imaginary time Schrödinger equation emerges from a Born-Oppenheimer approximation, and use this result to calculate the shift in the spectrum of radiated particles as a result of the tunneling. It is shown that the initial spectrum is shifted to favor low energy excitations, as is understood by realizing that the probability to tunnel is increased if energy is transferred from the radiation to the mirror.

In Sect. (3) this approach is extended to include gravity in asymptotically flat space. A WKB approximation to the Wheeler-DeWitt equation, as considered in [41, 42], is used to obtain an imaginary time Schrödinger equation which can then be solved as before. In Sect. (4) we use this result to examine the radiation from a black hole which is formed by tunneling. In particular, we consider the tunneling of a false vacuum bubble, a system extensively studied in Refs. [43] — [47]. This example involves a complication due to the peculiar structure that arises; Refs. [48, 18] show that the sequence of three-geometries encountered during tunneling can not be stacked together to form a manifold. Employing a slight modification of the standard approach, we show how the behaviour of fields on the Euclidean Schwarzschild manifold protects the late time radiation from being affected by tunneling. An intuitive reason for this is that the bubble's tunneling probability is unchanged by the presence of Hawking radiation, which involves the creation of pairs of particles with zero total energy.

6.1 Tunneling Mirror

Consider a mirror moving in a one dimensional potential in the presence of a massless scalar field. The Schrödinger equation for this system is

$$[\hat{H}_m + \hat{H}_\phi]\Psi[\phi, x_m; t] = i\frac{\partial}{\partial t}\Psi[\phi, x_m; t] \tag{6.1}$$

where

$$\hat{H}_m = -\frac{1}{2m}\frac{\partial^2}{\partial x_m^2} + V(x_m) \tag{6.2}$$

and

$$\hat{H}_\phi = \frac{1}{2}\int_{x_m}^{\infty} dx \left[-\frac{\delta^2}{\delta\phi(x)^2} + \left(\frac{d\phi}{dx}\right)^2\right]. \tag{6.3}$$

Note that Ψ is a function of the mirror coordinate x_m, and a functional of the field configuration $\phi(x)$. The mirror boundary condition is imposed by demanding that the field vanish at x_m,

$$\Psi[\phi, x_m; t] = 0 \quad \text{if} \quad \phi(x_m) \neq 0. \tag{6.4}$$

The system is solved by assuming that the backreaction of the field on the mirror is a small perturbation of the mirror's motion, and that the mass and momenta of the mirror are large enough that it can be described by a well localized wave packet. In this domain the system admits a Born-Oppenheimer approximation, which amounts to an expansion in $1/m$. In particular, we seek a solution to the time independent Schrödinger equation

$$[\hat{H}_m + \hat{H}_\phi]\Psi[\phi, x_m] = E\Psi[\phi, x_m] \tag{6.5}$$

valid to zeroth order in $1/m$. Following Refs. [38, 42] the Born-Oppenheimer approximation is implemented by writing Ψ in the form

$$\Psi[\phi, x_m] = \psi_{VV}(x_m)\, e^{iS(x_m)}\, \chi[\phi, x_m] \tag{6.6}$$

where ψ_{VV} is a slowly varying function to be identified with the Van Vleck determinant. To lowest order in $1/m$, (6.5) reduces to the Hamilton-Jacobi equation.

$$\frac{1}{2m}\left(\frac{dS}{dx_m}\right)^2 + V(x_m) = E \tag{6.7}$$

since dS/dx_m, $V(x_m)$ and E are all of order m.

To zeroth order:

$$-\frac{i}{2m}\frac{d^2S}{dx_m^2}\,\psi_{VV}\,\chi[\phi, x_m] - \frac{i}{m}\frac{dS}{dx_m}\frac{d\psi_{VV}}{dx_m}\chi[\phi, x_m] \tag{6.8}$$

$$-\frac{i}{m}\psi_{VV}\frac{dS}{dx_m}\frac{\partial}{\partial x_m}\chi[\phi, x_m] + \psi_{VV}\,\hat{H}_\phi\,\chi[\phi, x_m] = 0.$$

ψ_{VV} is chosen so that the first two terms cancel, leaving

$$\hat{H}_\phi\,\chi[\phi,x_m]=\frac{i}{m}\frac{dS}{dx_m}\frac{\partial}{\partial x_m}\chi[\phi,x_m]. \tag{6.9}$$

This can be put in a familiar form by defining the time variable $\tau(x_m)$. In a classically allowed region, where $E-V(x_m)>0$ and dS/dx_m is real, τ is defined by

$$\frac{d\tau}{dx_m}=\frac{m}{dS/dx_m}\quad \text{allowed regions} \tag{6.10}$$

whereas in a classically forbidden region with dS/dx_m imaginary,

$$\frac{d\tau_E}{dx_m}=i\frac{m}{dS/dx_m}\quad \text{forbidden regions.} \tag{6.11}$$

The resulting zeroth order equations for ϕ are:

$$\hat{H}_\phi\,\chi[\phi,\tau]=i\frac{\partial}{\partial\tau}\chi[\phi,\tau]\quad \text{allowed regions} \tag{6.12}$$

$$-\hat{H}_\phi\,\chi[\phi,\tau_E]=\frac{\partial}{\partial\tau_E}\chi[\phi,\tau_E]\quad \text{forbidden regions.} \tag{6.13}$$

These are the fundamental equations governing the evolution of the scalar field in the presence of the mirror. In the allowed regions we have recovered the time-dependent Schrödinger equation with the postion of the mirror playing the role of a clock, whereas in the forbidden regions we have obtained a diffusion equation, which we shall refer to as the Euclidean Schrödinger equation, with the Euclidean time τ_E measuring the position of the mirror in the potential barrier.

Now, choose the potential to be of the form illustrated in Fig. 6.1 and let the mirror come from right to left. In the allowed region to the right of x_m^i the state $\chi[\phi,\tau]$ obeys the normal Schrödinger equation, and so standard methods can be used to find $\chi[\phi,\tau^i]$. Between x_m^i and x_m^f the mirror is in a forbidden region, so the state evolves according to

$$-\frac{1}{2}\int_{x_m(\tau_E)}^{\infty}dx\left[-\frac{\delta^2}{\delta\phi(x)^2}+\left(\frac{d\phi}{dx}\right)^2\right]\chi[\phi,\tau_E]=\frac{\partial}{\partial\tau_E}\chi[\phi,\tau_E] \tag{6.14}$$

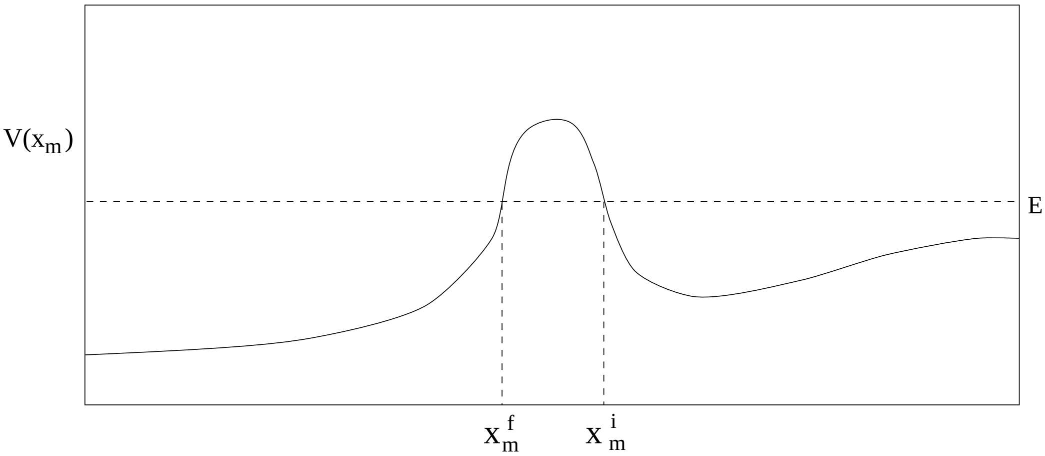

Figure 6.1: A generic mirror potential. The turning points for energy E are indicated.

with $\chi[\phi,\tau_E^i] = \chi[\phi,\tau^i]$. We wish to solve this equation in order to find the state at the final turning point x_m^f. It is useful to transform the mirror to rest by defining the coordinate

$$y(x,\tau_E) = x - x_m(\tau_E) \tag{6.15}$$

in terms of which the Euclidean Schrödinger equation is

$$-\frac{1}{2}\int_0^\infty dy\left[-\frac{\delta^2}{\delta\phi(y)^2} + 2\,\frac{dx_m}{d\tau_E}\frac{d\phi}{dy}\frac{\delta}{\delta\phi(y)} + \left(\frac{d\phi}{dy}\right)^2\right]\chi[\phi,\tau_E] = \frac{\partial}{\partial\tau_E}\chi[\phi,\tau_E] \tag{6.16}$$

or

$$-\hat{H}_\phi^E(\tau_E)\chi[\phi,\tau_E] = \frac{\partial}{\partial\tau_E}\chi[\phi,\tau_E]. \tag{6.17}$$

The solution is

$$\chi[\phi,\tau_E] = T\exp\left[-\int_{\tau_E^i}^{\tau_E}\hat{H}_\phi^E(\tau_E^{'})d\tau_E^{'}\right]\chi[\phi,\tau_E^i] = \hat{U}_E(\tau_E,\tau_E^i)\,\chi[\phi,\tau_E^i]. \tag{6.18}$$

Here T represents time ordering with respect to $\tau_E^{'}$. The crucial point is that the Euclidean time evolution operator, $\hat{U}_E$, is non-unitary. This is natural since we know that wavefunctions decay exponentially during tunneling. If

$\hat{U}_E$ was unitary, the easiest way to calculate it would be to transform to the Heisenberg picture, solve the field equations mode by mode, and compute Bogolubov coefficients. However, as emphasized in Ref. [38] the non-unitarity of $\hat{U}_E$ implies that the Schrödinger and Heisenberg pictures are inequivalent, making the standard method inapplicable. Instead, one can use the method developed in Ref. [38] which closely resembles the standard one but is more general. We first describe the state right before tunneling. For convenience, set $x_m^i = \tau^i = \tau_E^i = 0$. Let $\xi_\omega(x,\tau)$ be a complete set of positive norm solutions to the Klein-Gordon equation which vanish vanish at the mirror:

$$\left[-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial x^2}\right]\xi_\omega(x,\tau) = 0 \tag{6.19}$$

$$i\int dx\left[\xi_\omega^*(x,\tau)\,\frac{\partial}{\partial\tau}\xi_{\omega'}(x,\tau) - \frac{\partial}{\partial\tau}\xi_\omega^*(x,\tau)\xi_{\omega'}(x,\tau)\right] = \delta_{\omega\omega'} \tag{6.20}$$

$$\xi_\omega(x_m(\tau),\tau) = 0. \tag{6.21}$$

The set of allowed frequencies ω is taken to be discrete, and $\sum_\omega$ represents summation over this set. The field operators can then be expanded in terms of these modes:

$$\hat{\phi}(x,\tau) = \sum_\omega\left[\hat{a}_\omega\xi_\omega(x,\tau) + \hat{a}_\omega^\dagger\xi_\omega^*(x,\tau)\right] \tag{6.22}$$

$$\hat{\pi}_\phi(x,\tau) = \frac{\partial}{\partial\tau}\hat{\phi}(x,\tau) = \sum_\omega\left[\hat{a}_\omega\frac{\partial}{\partial\tau}\xi_\omega(x,\tau) + \hat{a}_\omega^\dagger\frac{\partial}{\partial\tau}\xi_\omega(x,\tau)\right] \tag{6.23}$$

with $\left[\hat{a}_\omega, \hat{a}_{\omega'}^\dagger\right] = \delta_{\omega\omega'}$.

Now define Euclidean fields $\hat{\phi}^E(y,\tau_E)$, $\hat{\pi}_\phi^E(y,\tau_E)$ which agree with $\hat{\phi}(x,\tau)$, $\hat{\pi}_\phi(y,\tau)$ at $\tau = \tau^E = 0$, but evolve according to

$$\hat{\phi}^E(y,\tau_E) = \hat{U}_E^{-1}(\tau_E,0)\,\hat{\phi}^E(y,0)\,\hat{U}_E(\tau_E,0) \tag{6.24}$$

$$\hat{\pi}_\phi^E(y,\tau_E) = \hat{U}_E^{-1}(\tau_E,0)\,\hat{\pi}_\phi^E(y,0)\,\hat{U}_E(\tau_E,0). \tag{6.25}$$

We will calculate $\hat{U}_E(\tau_E,0)$ by first finding $\hat{\phi}^E(y,\tau_E)$, $\hat{\pi}_\phi^E(y,\tau_E)$. The field equations for these operators are

$$\frac{\partial\hat{\phi}^E}{\partial\tau_E} = -\left[\hat{\phi}^E, \hat{H}_\phi^E\right] = -i\hat{\pi}_\phi^E + \frac{dx_m}{d\tau_E}\frac{\partial\hat{\phi}^E}{\partial y} \tag{6.26}$$

$$\frac{\partial \hat{\pi}_\phi^E}{\partial \tau_E} = -\left[\hat{\pi}_\phi^E, \hat{H}_\phi^E\right] = -i\frac{\partial^2 \hat{\phi}^E}{\partial y^2} + \frac{dx_m}{d\tau_E}\frac{\partial \hat{\pi}_\phi^E}{\partial y}. \tag{6.27}$$

So

$$\hat{\pi}_\phi^E = i\left(\frac{\partial \hat{\phi}^E}{\partial \tau_E} - \frac{dx_m}{d\tau_E}\frac{\partial \hat{\phi}^E}{\partial y}\right) \tag{6.28}$$

and

$$\frac{\partial^2 \hat{\phi}^E}{\partial \tau_E^2} + \left[1 + \left(\frac{dx_m}{d\tau_E}\right)^2\right]\frac{\partial^2 \hat{\phi}^E}{\partial y^2} - 2\frac{dx_m}{d\tau_E}\frac{\partial^2 \hat{\phi}^E}{\partial y \partial \tau_E} - \frac{d^2 x_m}{d\tau_E^2}\frac{\partial \hat{\phi}^E}{\partial y} = 0. \tag{6.29}$$

Equation (6.29) can be obtained by varying the action

$$S = \frac{1}{2}\int dy\, d\tau_E\, \sqrt{g_E}\, g_E^{\mu\nu}\partial_\mu \phi \partial_\nu \phi \tag{6.30}$$

with the Euclidean metric

$$ds_E^2 = g_{\mu\nu}^E dx^\mu dx^\nu = d\tau_E^2 + 2\,\frac{dx_m}{d\tau_E}\, dx\, d\tau_E + dx^{\,2}. \tag{6.31}$$

$\hat{\phi}^E$, $\hat{\pi}_\phi^E$ can be expanded in terms of modes f_ω which satisfy the Euclidean Klein-Gordon equation (6.29) and which vanish at $y = 0$,

$$\hat{\phi}^E(y, \tau_E) = \sum_\omega \hat{b}_\omega f_\omega(y, \tau_E) \tag{6.32}$$

$$\hat{\pi}_\phi^E(y, \tau_E) = i\sum_\omega \hat{b}_\omega \left(\frac{\partial}{\partial \tau_E} f_\omega(y, \tau_E) - \frac{dx_m}{d\tau_E}\frac{\partial}{\partial y} f_\omega(y, \tau_E)\right). \tag{6.33}$$

As the Euclidean Klein-Gordon equation is elliptic, one cannot in general impose Cauchy boundary conditions at $\tau_E = 0$ on f_ω. The resulting solutions would not satisfy the mirror boundary condition. With the appropriate boundary conditions, either Dirichlet or Neumann, imposed at $\tau_E = 0$ and $\tau_E = \tau_E^f$, a detailed calculation is, of course, required to find f_ω for a generic mirror trajectory. We shall take the solutions as given and only use their specific forms in a region far from the mirror, where they are simple.

Now, using the condition that the two sets of operators $\hat{\phi}$, $\hat{\pi}_\phi$ and $\hat{\phi}^E$, $\hat{\pi}_\phi^E$ are equal at $\tau = \tau_E = 0$, and taking inner products, the operators $\hat{b}_\omega$ can be expressed as a linear combination of $\hat{a}_\omega$, $\hat{a}_\omega^\dagger$:

$$\hat{b}_\omega = \sum_{\omega'}\left[\alpha_{\omega\omega'}\hat{a}_{\omega'} + \beta_{\omega\omega'}\hat{a}_{\omega'}^\dagger\right]. \tag{6.34}$$

Then using

$$\hat{\phi}^E(y,\tau_E^f) = \hat{U}_E^{-1}(\tau_E^f,0)\,\hat{\phi}_E(y,0)\,\hat{U}_E(\tau_E^f,0) = \hat{U}_E^{-1}(\tau_E^f,0)\,\hat{\phi}(y,0)\,\hat{U}_E(\tau_E^f,0) \tag{6.35}$$

and the analogous expression for $\hat{\pi}_\phi^E$, the following equations for $\hat{U}^E$ are obtained:

$$\sum_{\omega}\sum_{\omega'}\left[\alpha_{\omega\omega'}\hat{a}_{\omega'}+\beta_{\omega\omega'}\hat{a}^{\dagger}_{\omega'}\right]f_{\omega}(y,\tau^{f}_{E})$$
$$=\sum_{\omega}\left[\hat{U}_E^{-1}(\tau_E^f,0)\,\hat{a}_{\omega}\,\hat{U}_E(\tau_E^f,0)\,\xi_{\omega}(y,0)+\hat{U}_E^{-1}(\tau_E^f,0)\,\hat{a}^{\dagger}_{\omega}\,\hat{U}_E(\tau_E^f,0)\,\xi^{*}_{\omega}(y,0)\right] \tag{6.36}$$

and

$$i\sum_{\omega}\sum_{\omega'}\left[\alpha_{\omega\omega'}\hat{a}_{\omega'}+\beta_{\omega\omega'}\hat{a}^{\dagger}_{\omega'}\right]\frac{\partial}{\partial\tau_E}f_{\omega}(y,\tau^{f}_{E})$$
$$=\sum_{\omega}\left[\hat{U}_E^{-1}(\tau_E^f,0)\,\hat{a}_{\omega}\,\hat{U}_E(\tau_E^f,0)\,\frac{\partial}{\partial\tau}\xi_{\omega}(y,0)+\hat{U}_E^{-1}(\tau_E^f,0)\,\hat{a}^{\dagger}_{\omega}\,\hat{U}_E(\tau_E^f,0)\,\frac{\partial}{\partial\tau}\xi^{*}_{\omega}(y,0)\right]. \tag{6.37}$$

Again taking inner products, this leads to relations of the form

$$\hat{U}_E^{-1}(\tau_E^f,0)\,\hat{a}_{\omega}\,\hat{U}_E(\tau_E^f,0)=\sum_{\omega'}\left[u_{\omega\omega'}\hat{a}_{\omega'}+v_{\omega\omega'}\hat{a}^{\dagger}_{\omega'}\right] \tag{6.38}$$

$$\hat{U}_E^{-1}(\tau_E^f,0)\,\hat{a}^{\dagger}_{\omega}\,\hat{U}_E(\tau_E^f,0)=\sum_{\omega'}\left[w_{\omega\omega'}\hat{a}_{\omega'}+z_{\omega\omega'}\hat{a}_{\omega'}\right]. \tag{6.39}$$

Then it can be shown that [38]

$$\hat{U}_E(\tau_E^f,0)=\text{ const. }\times:\exp\sum_{\omega}\sum_{\omega'}\left[\frac{1}{2}D_{\omega\omega'}\hat{a}^{\dagger}_{\omega}\hat{a}^{\dagger}_{\omega'}+F_{\omega\omega'}\hat{a}_{\omega}\hat{a}_{\omega'}+\frac{1}{2}G_{\omega\omega'}\hat{a}_{\omega}\hat{a}_{\omega'}\right]: \tag{6.40}$$

where the matrices D, F, and G are defined by

$$D=vz^{-1}\ \ ;\ \ F=\left(z^{T}\right)^{-1}-1\ \ ;\ \ G=-z^{-1}w. \tag{6.41}$$

The state after tunneling is then determined,

$$\left|\chi(\tau_E^f)\right\rangle=\hat{U}_E(\tau_E^f)\left|\chi(0)\right\rangle \tag{6.42}$$

and is expressed in terms of occupation numbers with respect to the modes $\xi_{\omega}(y,0)$, where now $y=x-x_m^f$. All of the information about the final state is contained in the matrices D, F, and G, which are in turn given in terms of inner products between the modes f_{ω} and ξ_{ω}.

As a simple application of these formulæ we will calculate the shift in the spectrum of outgoing particles which are far from the mirror at the time of tunneling. It is assumed that the mirror was initially at rest and the field in its ground state. The mirror subsequently accelerates in the potential $V(x_m)$ until it reaches the classical turning point x_m^i. It is well known that as a result of the mirror's acceleration, a flux of outgoing particles is created whose spectrum is calculable by standard methods [28]. Outgoing particles far from the mirror are wavepackets composed of superpositions of plane waves,

$$\xi_\omega(x,\tau) = \frac{1}{2\sqrt{\omega}}\, e^{-i\omega(\tau - x)} \tag{6.43}$$

The spectrum of outgoing particles located at $x = \bar{x} \gg \omega^{-1}$ at $\tau = 0$ is written as

$$\sum_{\{n_\omega\}} S_{\bar{x}}\left(\{n_\omega\}\right) |\{n_\omega\}\rangle \tag{6.44}$$

where $\{n_\omega\}$ is a set of occupation numbers and $S_{\bar{x}}\left(\{n_\omega\}\right)$ is the amplitude for the set to occur.

Far from the mirror, the modes f_ω are easy to calculate since the mirror boundary condition is irrelevant. They are of two types,

$$\begin{aligned} f_\omega^- &= \frac{1}{2\sqrt{\omega}}\, e^{-\omega\tau_E + i\omega x} = \frac{1}{2\sqrt{\omega}}\, e^{-\omega\tau_E + i\omega(y + x_m(\tau_E))} \\ f_\omega^+ &= \frac{1}{2\sqrt{\omega}}\, e^{\omega\tau_E + i\omega x} = \frac{1}{2\sqrt{\omega}}\, e^{\omega\tau_E + i\omega(y + x_m(\tau_E))} \end{aligned} \tag{6.45}$$

Then $\hat{\phi}$, $\hat{\pi}$ and $\hat{\phi}^E$, $\hat{\pi}_\phi^E$ are equal at $\tau = \tau_E = 0$ if

$$\hat{b}_\omega^- = \hat{a}_\omega \quad ; \quad \hat{b}_\omega^+ = \hat{a}_\omega^\dagger . \tag{6.46}$$

Equation (6.38) gives:

$$\begin{aligned} \hat{U}_E^{-1}(\tau_E^f, 0)\, \hat{a}_\omega\, \hat{U}_E(\tau_E^f), 0) &= e^{-\omega\tau_E^f + i\omega x_m^f}\, \hat{a}_\omega \\ \hat{U}_E^{-1}(\tau_E^f, 0)\, \hat{a}_\omega^\dagger\, \hat{U}_E(\tau_E^f, 0) &= e^{\omega\tau_E^f + i\omega x_m^f}\, \hat{a}_\omega^\dagger \end{aligned} \tag{6.47}$$

leading to

$$D = G = 0 \quad ; \quad F_{\omega\omega'} = \left(e^{-\omega\tau_E^f - i\omega x_m^f} - 1\right) \delta_{\omega\omega'} \tag{6.48}$$

and

$$\hat{U}_E(\tau_E^f, 0) = \text{ const. } \times : \exp \sum_\omega \left[e^{-\omega\tau_E^f - i\omega x_m^f} - 1\right] \hat{a}_\omega^\dagger \hat{a}_\omega :$$

$$= \text{const.} \times : \exp \sum_{\omega} \left[e^{-i\omega x_m^f} - 1 \right] \hat{a}_\omega^\dagger \hat{a}_\omega :: \exp \sum_{\omega} \left[e^{-\omega \tau_E^f} - 1 \right] \hat{a}_\omega^\dagger \hat{a}_\omega : \quad (6.49)$$

The first factor is a translation operator which expresses the state in terms of the modes $\xi_\omega(x,0)$ instead of $\xi_\omega(x + x_m^f, 0)$, and the second factor acts on a state $|\{n_\omega\}\rangle$ to give $e^{-E(\{n_\omega\})\tau_E^f} |\{n_\omega\}\rangle$, where $E(\{n_\omega\}) = \sum n_\omega \omega$ is the energy of the state. Therefore, the state after tunneling is

$$\text{const.} \times \sum_{\{n_\omega\}} e^{-E(\{n_\omega\})\tau_E^f} S_{\bar{x}}(\{n_\omega\}) |\{n_\omega\}\rangle . \quad (6.50)$$

The result of the tunneling is simply to shift the spectrum from $S_{\bar{x}}$ to $e^{-E(n_\omega)\tau_E^f} S_{\bar{x}}$.

It is not difficult to understand this result. Since the total energy is fixed , the state before tunneling is given by a superpostion of the various ways of distributing the energy between the mirror and the radiation. As the mirror's probability to tunnel depends exponentially on its energy, we expect an inverse exponential correlation between tunneling and energy in radiation. Thus an observer measuring the spectrum of radiation, conditional on the mirror tunneling, finds the result (6.50). Far from the mirror the shift in the spectrum depends only on τ_E^f, the amount of Euclidean time spent during tunneling. This is because the tunneling amplitude in the WKB approximation is e^{-S}, and the derivative of S with respect to energy is just the Euclidean time.

If we were to identify the Euclidean time with an inverse temperature, the shift would become a Boltzmann factor. This makes it easy to generate thermal distributions of radiation. Specifically, if the distribution before tunneling was a constant, then after tunneling tracing over the states of the mirror would yield a thermal density matrix for the radiation. A number of authors have been led by this fact to seek a connection between the thermal radiation that arises in the contexts of cosmology and black holes and an occurrence of tunneling [40, 50, 51]. Such a connection relies upon assumptions about what is on the other side of the barrier and what the spectrum of radiation is there. In this work we only consider situations where there is a well defined classical trajectory on either side of the barrier; we are interested in the case in which there is collapsing matter on side of the barrier and a black hole on the other. The treatment of this process requires an extension of the previous method to include gravity.

6.2 Application to Gravity

In this section we make a WKB approximation to gravity in a manner which directly parallels that for the moving mirror. The action for gravity plus matter takes the form

$$S = \frac{m_p^2}{16\pi}\int d^4x\sqrt{-g}\,(\mathcal{R}-2\Lambda) + S_M + \text{ boundary terms}$$
$$= \int d^4x\left(\pi_{\phi_i}\dot{\phi}^i + \pi_{ij}\dot{h}^{ij} - N^t\mathcal{H}_t - N_i\mathcal{H}^i\right) + \text{ boundary terms.} \qquad (6.51)$$

The Wheeler-DeWitt equation resulting from this action is

$$\hat{\mathcal{H}}_t\Psi = \left[-\frac{16\pi}{m_p^2}G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \frac{m_p^2}{16\pi}h^{\frac{1}{2}}\left({}^3\mathcal{R}-2\Lambda\right) + \hat{\mathcal{H}}_{t_M}\right]\Psi = 0 \qquad (6.52)$$

Proceeding as before, we seek a semiclassical solution of the form

$$\Psi\left[h_{ij},\phi_i\right] = \psi_{VV}[h_{ij}]\,e^{im_p^2 S[h_{ij}]}\,\chi\left[\phi_i,h_{ij}\right]. \qquad (6.53)$$

At first order the Einstein-Hamilton-Jacobi equation is obtained:

$$\frac{16\pi}{m_p^2}G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} - \frac{m_p^2}{16\pi}h^{\frac{1}{2}}\left({}^3\mathcal{R}-2\Lambda\right) = 0. \qquad (6.54)$$

Zeroth order yields

$$-\frac{16\pi}{m_p^2}i\,G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta\chi}{\delta h_{kl}} + \hat{\mathcal{H}}_{t_M}\chi = 0 \qquad (6.55)$$

provided ψ_{VV} satisfies

$$G_{ijkl}\frac{\delta^2 S}{\delta h_{ij}\delta h_{ij}}\psi_{VV} + G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta\psi_{VV}}{\delta h_{kl}} = 0. \qquad (6.56)$$

The momentum constraints at first order are

$$\left(\frac{\delta S}{\delta h_{ij}}\right)_{|j} = 0 \qquad (6.57)$$

and at zeroth order are

$$2i\left(\frac{\delta\chi}{\delta h_{ij}}\right)_{|j} + \hat{\mathcal{H}}^i_M\,\chi = 0. \tag{6.58}$$

Equations (6.55) and (6.58) describe how the matter wave function evolves as the spatial geometry changes. Quantum field theory in curved space can be recovered by writing χ's dependence on h_{ij} in terms of a time functional $\tau[x;h_{ij}]$, and by reintroducing a lapse N^τ and shift N_i, demanding that they obey

$$G_{ijkl}\frac{\delta S}{\delta h_{ij}} = \frac{m_p^2}{16\pi N^\tau}\left(\int dy \frac{\delta h_{kl}}{\delta\tau[y;h_{ab}]} - N_{i|j} - N_{j|i}\right). \tag{6.59}$$

Then

$$-i\frac{16\pi}{m_p^2}\int\left(N^\tau G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta\chi}{\delta h_{kl}} + 2iN_i\left(\frac{\delta\chi}{\delta h_{ij}}\right)_{|j}\right) = -i\int\frac{\delta h_{ij}}{\delta\tau}\frac{\delta\chi}{\delta h_{ij}} \tag{6.60}$$

so that the equation for χ becomes

$$\int d^3x\left[N^\tau\hat{\mathcal{H}}_{t_M} + N_i\hat{\mathcal{H}}^i_M\right]\chi[\phi_i;\tau] = i\frac{\partial}{\partial\tau}\chi[\phi_i;\tau]. \tag{6.61}$$

The condition (6.59) agrees with the classical relation between π_{ij} and h_{ij}, demonstrating that $\tau[x;h_{ij}]$ is the classical time and that (6.61) is the Schrödinger picture version of quantum field theory in curved space.

As with the mirror example, τ becomes imaginary during tunneling so we define a Euclidean time τ_E along with a Euclidean lapse $N^{\tau_E} = iN^\tau$, in terms of which χ obeys

$$-\int dx[N^{\tau_E}\hat{\mathcal{H}}_{t_M} + iN_i\hat{\mathcal{H}}^i_M]\chi[\phi_i,\tau_E] = \frac{\partial}{\partial\tau_E}\chi[\phi_i,\tau_E]. \tag{6.62}$$

For a massless scalar field with action

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\,g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi, \tag{6.63}$$

we have

$$\hat{\mathcal{H}}_{t_M} = \frac{1}{2}\left(h^{-\frac{1}{2}}\hat{\pi}_\phi^2 + h^{\frac{1}{2}}h^{ij}\partial_i\hat{\phi}\,\partial_j\hat{\phi}\right) \tag{6.64}$$

$$\hat{\mathcal{H}}_{i_M} = \partial_i \hat{\phi}\, \hat{\pi}_\phi. \tag{6.65}$$

To evolve χ through the tunneling region one is required to calculate the Euclidean time evolution operator

$$\hat{U}_E(\tau_E^f, \tau_E^i) = T \exp\left[-\int_{\tau_E^i}^{\tau_E^f} \hat{\mathcal{H}}_\phi^E d\tau_E\right] \tag{6.66}$$

with

$$\hat{H}_\phi^E = \int d^3x\, [N^{\tau_E}(-\frac{1}{2}h^{-\frac{1}{2}}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}h^{\frac{1}{2}}h^{ij}\partial_i\phi\,\partial_j\phi) + N_i\partial_i\phi\frac{\delta}{\delta\phi}]. \tag{6.67}$$

As before, one proceeds by defining Euclidean fields obeying (6.24,6.25). In the present case the resulting field equations are:

$$\left(\sqrt{g_E}\, g_E^{\mu\nu}\partial_\mu\hat{\phi}^E\right)_{,\nu} = 0$$

$$\hat{\pi}_\phi^E = i\frac{h^{\frac{1}{2}}}{N^{\tau_E}}\left(\frac{\partial\hat{\phi}_E}{\partial\tau_E} - N^i\partial_i\phi\right) \tag{6.68}$$

with

$$ds_E^2 = g_{\mu\nu}^E\, dx^\mu dx^\nu = (N^{\tau_E}d\tau_E)^2 + h_{ij}\left(dx^i + N_i d\tau_E\right)\left(dx^j + N^j d\tau_E\right). \tag{6.69}$$

The evolution operator, and therefore the state after tunneling, is determined by solving the field equations mode by mode, and repeating the steps leading from (6.32) to (6.42).

6.3 Black Hole Radiance in the Presence of Tunneling

We can now apply this method to determine how the radiation from a black hole is affected by tunneling. It is well known that a black hole formed classically from collapsing matter radiates in a complicated manner at early times due to the time dependent geometry, but at late times will inevitably radiate as a black body at the Hawking temperature. Is this scenario altered if the black hole is formed while tunneling? We shall show that it is not. The form of the late time radiation is insensitive to the hole's unconventional history in a way that is consistent with the intuitive picture of Hawking radiation being caused by pair production near the horizon.

We consider the behaviour of a scalar field on the background of a false vacuum bubble which tunnels leading to the formation of a black hole. The action for a false vacuum bubble in the thin wall approximation is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,\mathcal{R} - \frac{\Lambda_I}{8\pi}\int_{bubble} d^4x\sqrt{-g} - \frac{\mu}{4\pi}\int_{wall} d^3A \tag{6.70}$$

where Λ_I is the cosmological constant of the false vacuum, and μ is the energy density of the bubble wall. The classical solutions for this action have been derived in Refs. [43]-[47]. In what follows we refer to the treatment of Ref. [47]. The spherically symmetric solutions are characterized by three parameters: Λ_I, μ, and the total mass M. In addition, for given Λ_I and μ there is a critical mass M_{cr} below which there are two solutions: type (a), where the bubble emerges from a singularity with zero radius, subsequently expands to a maximum radius, and then recollapses; type (b), where the bubble initially collapses from infinite radius, reaches a minimum radius, and then reexpands. Using the results of Refs. [48, 18], we focus on an expanding solution of type (a) which tunnels to an expanding solution of type (b). We confine our interest to the region outside the bubble where the metric, written in terms of Schwarzschild time t and $r_* = r + 2M\ln(r/2M - 1)$, is

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2 d\Omega^2. \tag{6.71}$$

As t and r_* cover only part of the complete manifold, we introduce Kruskal-Szekeres coordinates,

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dX^2) + r^2 d\Omega^2. \tag{6.72}$$

The two sets of coordinates are related by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = X^2 - T^2$$

$$t = \begin{cases} 4M\tanh^{-1}(T/X) & \text{if} \quad |T/X| < 1 \\ 4M\tanh^{-1}(X/T) & \text{if} \quad |T/X| > 1 \end{cases} \tag{6.73}$$

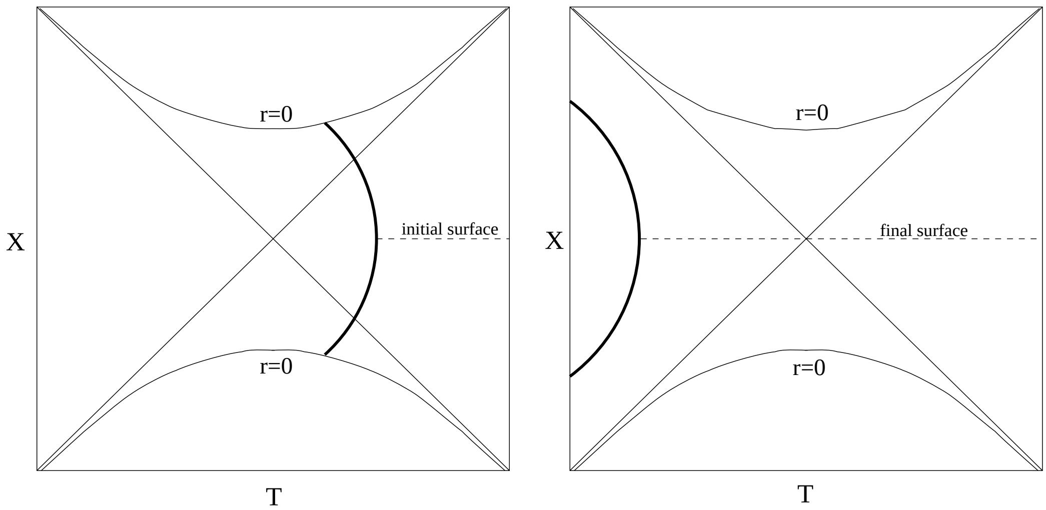

Figure 6.2: The type (a) and (b) solutions. The heavy lines represent the bubble trajectory, and the dashed lines are the initial and final surfaces of the tunneling solution. In these figures, only the regions to the right of the trajectory are of interest, as they are outside of the bubble.

Using these cordinates the type (a) and (b) solutions of interest are depicted in Fig. 6.2.

The tunneling amplitude for this process has been computed by two different methods. In Ref. [18] the solution to the Wheeler-DeWitt equation is found in the WKB approximation by solving the Einstein-Hamilton-Jacobi equation (6.54). Since the solution behaves as e^{-S}, and the tunneling amplitude is given by the ratio of the wavefunction evaluated at the initial and final geometries, the tunneling amplitude is

$$\exp\left(S[h_{ij}^{\text{initial}}] - S[h_{ij}^{\text{final}}]\right) \tag{6.74}$$

No difficulties arise in this approach; the calculation of tunneling amplitude proceeds in a straightforward fashion.

In Ref. [48] the calculation is performed using the functional integral. In this formalism one looks for a manifold which interpolates between the initial and final surfaces and which is a solution to the Euclidean Einstein equations. The tunneling amplitude is e^{-S}, where S is the action of the solution. It is found, however, that solving the field equations leads to a sequence of three geometries which do not form a manifold. To see this, first note that the geometry outside the bubble is Euclidean Schwarzschild space, obtained by $t \to it_E$, $T \to iT_E$,

$$ds_E^2 = \left(1 - \frac{2M}{r}\right)\left(dt_E^2 + dr_*^2\right) + r^2 d\Omega^2 = \frac{32M^3 e^{-r/2M}}{r}\left(dT^2 + dX^2\right) + r^2 d\Omega^2 \tag{6.75}$$

with

$$\left(\frac{r}{2M} - 1\right) e^{r/2m} = X^2 + T_E^2 \; ; \; t_E = 4M \tan^{-1}(T_E/X). \tag{6.76}$$

It remains to describe the motion of the bubble wall. Solving the equations of motion leads to the trajectory in Fig. 6.3.

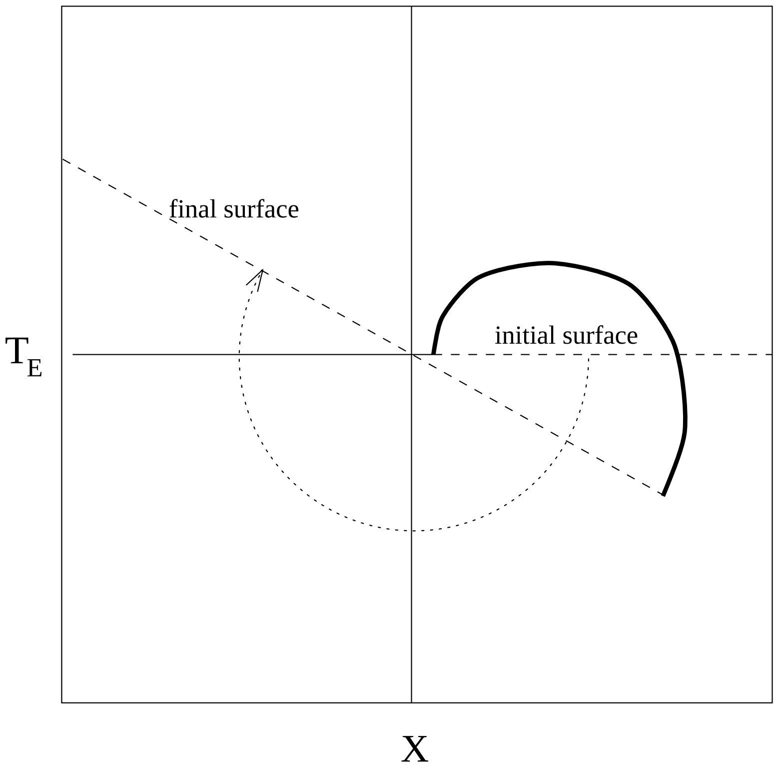

Figure 6.3: Bubble trajectory in Euclidean Schwarzschild space.

It is seen that the bubble wall crosses the initial surface during the course of its motion, creating a situation in which it is impossible to identify a region which is swept out by the evolving hypersurface. Some regions of the manifold are crossed twice by the hypersurface, some once, and some not at all. The authors of Ref. [48] call this object a pseudomanifold and give a prescription to calculate its action by assigning covering numbers to the various regions, but this is not needed for what follows.

With these results in hand, the technique of Sect. (3) can be used to calculate the state of the scalar field after tunneling. It was seen that once the solution of the Einstein-Hamilton-Jacobi equation is given, the field wave functional χ is fully determined by equations (6.55) and (6.58). Since $S[h_{ij}]$ is calculated in Ref. [18], we have all that we need to find χ. This would, however, require finding the solution to an unfamiliar functional differential equation. To cast it in in the form of the Schrödinger equation a lapse N^{τ_E}, shift N_i and time τ_E were reintroduced leading to the appearance of the Euclidean metric $g^E_{\mu\nu}$. In the present case there is no true interpolating Euclidean manifold, so that any choice of N^{τ_E} and N_i which define a well behaved $g^E_{\mu\nu}$ will lead to a bubble trajectory that is a multivalued function of time. Alternatively, a choice of time functional which gives a single valued bubble trajectory will necessarily lead to a Euclidean metric with vanishing determinant at some point. In either case, it is not clear that the resulting Schrödinger equation is well defined. This is apparent from Fig. 6.3, where it can be seen that boundary conditions imposed on the initial surface and on the bubble wall may contradict each other. These difficulties arise as a result of trying to compute the final state of the field in one step, which requires a Euclidean manifold interpolating all the way from the initial surface to the final surface, and can be avoided by calculating the state on a series of intermediate hypersurfaces. In this approach, it does not matter that the bubble wall eventually crosses the initial surface since once the state is calculated at some intermediate point we can forget about what preceded it.

For simplicity, we will consider only the s-wave component of the scalar field and frequencies high enough such that the geometrical optics approximation is valid. This means that the field equation is taken to be

$$g_E^{\mu\nu}\partial_\mu\partial_\nu\phi = 0. \tag{6.77}$$

The state of the field on the initial surface, $t = T = 0$, is most conveniently expressed in terms of the coordinates r_* and t. We divide the modes into ingoing and outgoing,

$$\xi_\omega^{\rm in}(r_*,t) = C_\omega\, e^{-i\omega(t+r_*)}$$
$$\xi_\omega^{\rm out}(r_*,t) = C_\omega\, e^{-i\omega(t-r_*)} \qquad (6.78)$$

and write the field operator as

$$\hat{\phi}(r_*,t) = \sum_\omega \left[\hat{a}_\omega^{\rm in}\xi_\omega^{\rm in} + \hat{a}_\omega^{\rm in\dagger}\xi_\omega^{\rm in*}(r_*,t) + \text{ in} \to \text{out}\right]. \qquad (6.79)$$

C_ω are normalization constants whose values will not be important. We shall only consider the *in* modes as the treatment of the *out* modes is exactly the same. We also suppress the *in* superscript.

In the first stage of the evolution the hypersurface is pivoted around $r_* = r_*^b$ by 180°, where r_*^b is the position of the bubble wall on the initial surface. The solutions to the Euclidean field equations are most conveniently obtained by choosing Cauchy boundary conditions on the initial surface, (clearly a valid procedure in this case)

$$f_\omega^+(r_*,0) = \xi_\omega(r_*,0) \;\; ; \;\; \frac{\partial}{\partial t_E} f_\omega^+(r_*,0) = -i\frac{\partial}{\partial t}\xi_\omega^*(r_*,0)$$
$$f_\omega^-(r_*,0) = \xi_\omega^*(r_*,0) \;\; ; \;\; \frac{\partial}{\partial t_E} f_\omega^-(r_*,0) = -i\frac{\partial}{\partial t}\xi_\omega^*(r_*,0) \qquad (6.80)$$

It is also easiest to use the X, T coordinates as they are well behaved everywhere. Since the evolution of the hypersurface is simply a reflection about the point $X = X^b$, a mode which has the form $f(X,T_E)$ on the initial surface has the form $f(-X+2X^b,T)$ on the new surface. Using the relations

$$r_* = 4M\ln\sqrt{X^2+T_E^2} \;\; ; \;\; t_E = 4M\tan^{-1}(T_E/X) \qquad (6.81)$$

and that

$$f_\omega^\pm(r_*,t_E) = C_\omega\, e^{\pm\omega t_E - i\omega r_*} \qquad (6.82)$$

near the initial surface, one sees that near the new surface,

$$f_\omega^\pm(X,T_E) = C_\omega \exp\left(\frac{\mp 4M\omega T_E}{-X+2X^b} - 4iM\omega\ln(-X+2X^b)\right). \qquad (6.83)$$

Since on the new surface, $f_\omega^+ = (f_\omega^-)^*$ and $\partial f_\omega^+/\partial t_E = -(\partial f_\omega^-/\partial t_E)^*$, the evolution operator $\hat{U}_E$ is unitary. This means that the state on the new surface has the same form as it did on the initial surface, but is now expressed in terms of the modes

$$\xi_\omega(X,T) = C_\omega \exp\left(\frac{-4iM\omega T}{-X+2X^b} + 4iM\omega\ln(-X+2X^b)\right). \tag{6.84}$$

These modes can be approximated near $T=0$ as

$$\xi_\omega = \begin{cases} C_\omega\, e^{i\omega(t-r_*)} & \text{if } \ |X| \gg X^b \\ C_\omega\, e^{-(2iM\omega/X^b)(T-X)} & \text{if } \ |X| \ll X^b \end{cases} \tag{6.85}$$

Now it is useful to express the state in terms of modes which are nonzero only inside or outside the horizon,

$$\eta_\omega^< = \begin{cases} D_\omega\, e^{i\omega(t-r_*)} & \text{if } \ X<0 \\ 0 & \text{if } \ X>0 \end{cases}$$

$$\eta_\omega^> = \begin{cases} 0 & \text{if } \ X<0 \\ D_\omega\, e^{-i\omega(t+r_*)} & \text{if } \ X>0. \end{cases} \tag{6.86}$$

A fundamental result [3, 26] in the derivation of black hole radiance is that the vacuum state with respect to modes which have a time dependence $e^{-i\omega T}$ is the state

$$\text{const.} \times \sum_{\{n_\omega\}} e^{-E(\{n_\omega\})/2T_H}\, |\{n_\omega\}\rangle_<\, |\{n_\omega\}\rangle_> \tag{6.87}$$

with respect to the modes $\eta_\omega^<$ and $\eta_\omega^>$. The sum runs over all sets of occupation numbers, $E=\sum n_\omega \omega$, and $T_H = 1/8\pi M$ is the Hawking temperature. Further, near the horizon, any deviation of $|\chi\rangle$ from the vacuum state can be ignored because of the arbitrarily large redshift as $r_* \to -\infty$. Far from the horizon ξ_ω and $\eta_\omega^<$ agree so the form of the state is unchanged there.

Now the hypersurface can be evolved the remainder of the way. If we restrict our attention to the region $X < X^b$, then the motion of the hypersurface is simply a translation, $t_E \to t_E - \Delta t_E$. This causes states with time dependence $e^{i\omega t}$ to be damped by a factor $e^{-\omega \Delta t_E}$, and states with time dependence $e^{-i\omega t}$ to be amplified by a factor $e^{\omega\Delta t_E}$. Near the horizon, the state $|\chi\rangle$ consists of pairs of positive and negative frequency states according to (6.87). One member of the pair is damped but the other is amplified by a compensating amount so as to leave the state $|\chi\rangle$ unchanged. The final state of the field can then be summarized as follows. Far from the hole, where there is no pairing, the initial state is damped:

$$\sum_{\{n_\omega\}} S(\{n_\omega\}) \, |\{n_\omega\}\rangle \longrightarrow \text{const.} \times \sum_{\{n_\omega\}} e^{-E(\{n_\omega\})\Delta t_E} \, S(\{n_\omega\}) \, |\{n_\omega\}\rangle \, . \quad (6.88)$$

Near the horizon the final state is given by (6.87). This is true for both the in and out modes, so an observer stationed on either side of the horizon would observe a thermal distribution of both ingoing and outgoing particles. As time passes, all of the ingoing particles will eventually cross the horizon and be swallowed by the hole, whereas the outgoing particles will propagate out to infinity where they can be detected at arbitarily late times as a flux of thermal radiation at the Hawking temperature.

6.4 Comments

It was shown that the standard picture of black hole radiance is unchanged by tunneling. At late times, the hole radiates just as it would have had it been formed from a classical collapse. This makes sense if one thinks of Hawking radiation as pair production. The probability of tunneling is not affected by the creation of a pair, since the pair has zero total energy. From this point of view it is also clear that what happens at early times cannot possibly affect the late time radiation, since the produced pairs only see the late time geometry. The conventional derivation of radiance obscures this point somewhat and it seems desirable to find an approach which makes this feature manifest from the outset. For the two systems considered here, and presumably this is true in general, the effect of the tunneling was to shift the distribution of any particles that were present before tunneling. In the present case initial excitations were damped because the final surface is rotated clockwise relative to the initial surface. A counterclockwise rotation would have led to amplification. In [48] numerical investigations are quoted which show that the rotation is always clockwise for the false vacuum bubble. One is led to speculate whether this is a general phenomenon — whether all tunneling transitions lead to damping.

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